

Poisson Structure Induced Field Theories and Models of 1 + 1 Dimensional Gravity

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Chapter 1

Introduction

One of the most prominent open problems in theoretical physics is to find some common understanding of the standard model on the one hand and Einsteins' theory of gravity on the other hand. Both theories have their own esthetic appeal, the former because it unified fundamental forces and the latter because of its geometric interpretation; both theories, furthermore, found sufficient experimental support within the realm of their validity. However, they are in conflict with each other: Within general relativity the matter system remains unquantized, whereas the standard model inherently is a quantum theory. The attempt to quantize gravity by means of the (perturbative) methods used successfully in the standard model failed thus far. This led to various alternative approaches. Beside string theory and noncommutative geometry the most prominent among these is the search for a consistent *nonperturbative* quantum theory of the coupled Einstein-Yang-Mills-matter system. Old hopes for success into this direction found some revival due to the pioneering works of Ashtekar [1], Rovelli and Smolin [2].

To get a better grasp on technical as well as conceptual problems encountered in this approach the study of the quantization of truncated versions of the full theory (Bianchi models) or, related to it, of lower dimensional models is suggestive. Whereas in three space-time dimensions the Einstein-Hilbert action

$$\int d^3x \sqrt{-g} R \tag{1.1}$$

for gravity is meaningful (here R denotes the torsionless Ricci scalar and g the determinant of the metric), in two space-time dimensions it yields no field equations, because it is a boundary term: $\int_M d\omega(e^a) = \int_{\partial M} \omega(e^a)$, where $\omega(e^a)$ is the torsionless spin connection. This led to the proposal of various other gravity actions in two dimensions.

One of these, proposed by Jackiw and Teitelboim [3], has the form

$$L^{JT} \propto \int_M d^2x \sqrt{-g} \Phi (R - \text{const}), \quad (1.2)$$

where Φ is some Lagrange multiplier field. Another action studied in two dimensions is

$$L^{R^2} = \int_M d^2x \sqrt{-g} (R^2/16 + \Lambda). \quad (1.3)$$

In contrast to (1.2) it is purely geometrical. It, however, leads to higher derivative equations of motion for the metric. Using for (1.3) Cartan variables, the torsion zero condition does not evolve as an equation of motion,¹ but it has to be implemented via a Lagrange multiplier or by expressing ω in terms of the zweibein e^a . The most natural Lagrangian for two-dimensional gravity when using Cartan variables was proposed by Katanaev and Volovich [4]:

$$L^{KV} = \int [-\frac{1}{4} d\omega \wedge *d\omega - \frac{1}{2\alpha} De^a \wedge *De_a + \Lambda \varepsilon]. \quad (1.4)$$

In two space-time dimensions this is the most general Lagrangian yielding second order differential equations for zweibein and spin-connection; it is purely geometrical and (but) torsion De^a became 'dynamical'. Another model of pure 2D gravity, gaining much interest recently, is defined by the string-inspired action [5]

$$L^{str} \propto \int_M d^2x \sqrt{-g} \exp(-2\Phi) [R + 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \Lambda] \quad (1.5)$$

where Φ is the Dilaton field.

The action (1.4) has some formal similarity with the one of a Yang-Mills theory for the Poincaré group. For $\Lambda = 0$ the only (but decisive) difference is

¹Contrary to what happens in the Palatini formulation of the four-dimensional Einstein-Hilbert action.

that the Hodge dual operation is not taken with respect to some background metric, but with part of the 'Poincaré connection', namely the zweibein, itself. Implementing the torsion zero condition in (1.2), on the other hand, through a Lagrange multiplier X_a , the Jackiw-Teitelboim (JT) model can be formulated equivalently as a connection flat gauge theory [6]

$$L = \int X^i F_i, \quad (1.6)$$

where $X^3 = \Phi$ and the F_i are the components of the $so(2,1)$ -curvature two-form corresponding to the connection ($a \in \{1,2\}$)

$$A_a \equiv e_a \quad A_3 \equiv \omega. \quad (1.7)$$

Similarly the action (1.1) was found to be *equivalent* to a $ISO(2,1)$ -Chern-Simons gauge theory [7]. Even the Ashtekar formulation of 4D gravity has some striking similarities (but also differences!) with a 4D (nonabelian) gauge theory, which are, e.g., the reason for the successful use of Wilson loops within the gravity theory [2].

Observations such as these and the partial success in finding the quantum theory for (1.1) and (1.2) due to the gauge theory formulations led some people to reinterpret the vielbein of any gravity theory as the 'missing' part of a Poincaré connection beside the spin (or Lorentz) connection; any gravity theory becomes a Poincaré gauge theory then by an appropriate introduction of additional auxiliary fields [8]. The flaw in this approach is that in general the diffeomorphism invariance, which is the main cause for the problems in the canonical quantization of gravity, remains still independent of the Poincaré gauge transformations (in contrast to what happens, e.g., with (1.1)). This becomes most obvious in the Hamiltonian formulation [9]. Still, the question remains: Can one draw (further) profit from the common structures of gravity and nonabelian gauge theories?

In two dimensions this question can be answered to the positive. The common structure between 2D Yang-Mills theories and (at least most) models of pure 2D gravity has a name: It is a Poisson structure P in the *target space* of the theory [10]. The first order action for these theories has the common

form²

$$\int_M A_i \wedge dX^i + \frac{1}{2} P^{ij}(X) A_i \wedge A_j. \quad (1.8)$$

Here $X^i(x)$ is the map from the space-time or worldsheet manifold M to the target space N , P is the Poisson tensor defined on the latter space, and A_i is a one-form on M . E.g., a Poisson structure linear in X yields gauge theories of the form (1.6); or the action (1.4) can be reproduced by the choice of a quadratic Poisson structure when integrating out X and making use of the identification (1.7).

The action (1.8) (and an appropriate extension of it) allows not only to study a large class of two-dimensional gravity as well as Yang-Mills theories at one and the same time and footing, the knowledge of having to deal with a Poisson structure on the space N suggests also the use of otherwise unusual kind of methods. In particular, diffeomorphisms in the target space can be used to bring P into some standard form generalizing the Darboux form of a nondegenerate P . In this way previous lengthy calculations can be reduced to some lines and the possibility to solve the incorporated theories on the quantum and classical level for all kind of different topologies seems close at hand.

To get a first feeling for the theory as defined in (1.8), let us use the field equations of the A_i to 'integrate them out' within this action. For simplicity we assume that $N \sim \mathbb{R}^n$ and that P is nondegenerate. P then has an inverse Ω , which is a symplectic two-form on N . Up to a multiplicative factor the action then takes the form

$$\int_M \Omega_{ij} dX^i dX^j. \quad (1.9)$$

This illustrates an important characteristic of the model: Since Ω is closed, (1.9) is a Wess-Zumino type action and thus, for finite n , (1.8) defines a model with only a *finite* number of (physical) degrees of freedom.³

²Independently of us the study of an action equivalent to (1.8) has been proposed also in [11].

³Recently I found the reference [12], where the action (1.9), arising from (1.8) for non-degenerate P , was studied from some other perspective and shown to be equivalent to

The organization of this report is as follows: In the next chapter we study (an extension of) the theory (1.8) in its own right. To not mix up the structures defined on M and N , we start with a study of Poisson structures defined on some finite dimensional manifold N . In the following section we then define the action providing details about its symmetry content, Hamiltonian and BRS formulation, etc. Thereafter we study the classical theory. Locally its integrability is basically trivial in this formulation. But part of the field equations are solved also for completely arbitrary topologies of M and N . The remaining equations of motion are, furthermore, particularly simple in an appropriate local coordinate system on N . In the concluding section of this chapter we then come to the quantum theory as defined on $M = S^1 \times \mathbb{R}$, such that one may use standard Hamiltonian methods. We construct *all* quantum states. Up to some technicalities of topological origin to be explained there, the 'physical' wave functionals are basically functions of a finite number of variables only, as expected already from (1.9).

The third and last chapter focuses on the gravity version of (1.8). First we find the most general class of models contained in this action which allows for a gravitational interpretation via (1.7). Restricting ourselves to a subclass of these, including all torsion-free ones, we show that locally the metric always can be brought into the 'generalized Schwarz-schild form'

$$g = h(r)(dt)^2 - \frac{(dr)^2}{h(r)}, \quad (1.10)$$

where *any* function h can be provided by an appropriate choice of the Lagrangian. We then solve the equations for the extremals in all generality and construct the universal covering solutions by means of Penrose diagrams. The considerations are illustrated at the examples of (1.2), (1.3), and (1.4) (cf. Figs. 5,6,8). On the quantum level the wave functions depend on one continuous parameter. Additional discrete labels of the wave functions arise in the case of a nontrivial causal structure of the classical theory, i.e. if h vanishes at some values of r .

Witten's topological sigma model on the quantum level (cf. also [13]). It will be interesting to further investigate (1.8) in view of this connection, also for degenerate Poisson structures.

In the remaining sections we take up three issues which might have their parallels also in the Ashtekar approach to quantum gravity. Firstly, the restriction to topologies of the form $\Sigma \times \mathbb{R}$, characteristic for any Hamiltonian treatment, is called into question. By means of the previously obtained Penrose diagrams we construct *all* global solutions for the example of the Katanaev-Volovich (KV) model (1.4) (with Minkowski signature). The space of these solutions is then compared to the reduced phase space RPS (\sim space of solutions on $M = S^1 \times \mathbb{R}$ modulo symmetry transformations) underlying the quantum theory. On the one hand the numbers of continuous and discrete parameters fit nicely, if we strictly stay with the cylindrical solutions. On the other hand, parts of the RPS are found to correspond to classical solutions for which some other topology, as, e.g., a torus with hole, would be more natural.

Secondly, (1.8) stays well-defined also for a configuration corresponding to a degenerate metric; furthermore, the Hamiltonian symmetries identify nondegenerate metrics with degenerate ones. This imitates somewhat the Ashtekar formulation of four dimensional gravity, which is also nonsingular at degenerate metrics [14]. For the two-dimensional models at hand, a detailed comparison of the standard Hamiltonian RPS with the one resulting from dividing out conventional gravity symmetries is possible. It reveals some inequivalence even after having excluded the nondegenerate solutions. The solutions identified in the Hamiltonian formulation differ by different kink number k . However, all solutions with $k \neq 0$ turn out to be geodesically incomplete.

Thirdly, we study the example of R^2 -gravity coupled to an $SU(2)$ -Yang Mills theory from the conceptual point of view. As in any quantum theory of gravity the Dirac observables are space-time independent and the Hamiltonian vanishes on physical quantum states. Strategies to resolve this apparent 'problem of (space-)time' [15] are developed at the example of the reparametrization invariant nonrelativistic particle. Realizing these strategies in the gravity-Yang-Mills system, one finds some partial confirmation of them through the fact that a gravity flat limit reproduces the usual $SU(2)$

quantum dynamics.

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The progress summarized in this work would have been unimaginable without the input of two valuable collaborators: Peter Schaller and Thomas Klösch. Let me thank them at this point. Sections 3.2, 3.3, and 3.5 will be published together with T. Klösch [16]. Practically all of the rest either will be or already has been published together with P. Schaller [17, 10, 18]. Works paving the way for the present account are further [19, 20, 21], beside the works of colleagues of mine at my institute and elsewhere [22, 23, 24]. Let me express also my gratitude to my supervisor Prof. W. Kummer for encouragement during the time of the thesis and to Prof. W. Thirring for his steady interest.

Chapter 2

Poisson Structure Induced Two Dimensional Field Theories

2.1 Poisson Structures and Symplectic Leaves

Let N denote a finite dimensional manifold and $\mathcal{F}(N)$ the space of smooth functions on it. A Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{F}(N)$ is a bilinear map $\mathcal{F}(N) \times \mathcal{F}(N) \rightarrow \mathcal{F}(N)$ which is skew-symmetric $\{F, G\} = -\{G, F\}$, obeys the Jacobi identity

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0, \quad (2.1)$$

and fulfills the Leibnitz rule: $\{F, GH\} = \{F, G\}H + G\{F, H\}$. Due to the latter requirement and the bilinearity any Poisson bracket can be represented by a (skew-symmetric) bivector field $P \in \Lambda^2(TN)$:

$$\{F, G\} = P(F, G) = P^{ij}(X) \frac{\partial F(X)}{\partial X^i} \frac{\partial G(X)}{\partial X^j}, \quad (2.2)$$

where we have chosen local coordinates $X^i, i = 1, \dots, n$ on N . The Jacobi identity becomes

$$P^{k[l} P^{ij]},_{k} = 0, \quad (2.3)$$

where $[...]$ denotes antisymmetrization and the comma a derivative. In more abstract terms, it becomes the vanishing of the Schouten-Nijenhuis bracket of P with itself. The latter bracket is a natural (graded) extension of the

Poisson bracket (resp. Lie bracket) to $\Lambda(TN) = \sum_{l=0}^n \Lambda^l(TN)$ (cf., e.g., [25], [26]).

A Poisson structure P is more general than a symplectic one since P need not be nondegenerate. Locally any Poisson structure $P \in \Lambda^2(TN)$ is characterized only by n , the dimension of the underlying manifold N , as well as the (local) dimension k of the kernel of P . An exception to this occurs for 'singular points' in N which are not part of any neighborhood with constant $\dim \ker P$.

Let us expand on this: The insertion of any one-form $e \in T^*M$ not in the kernel of P provides a vector field; the latter is called (locally) Hamiltonian, if (locally) $e = dF$ for some function $F \in \mathcal{F}(N)$. As a consequence of the Jacobi identity (2.3), the set of locally Hamiltonian vector fields is in involution. Thus, according to the Frobenius theorem, locally they generate an integral surface S through any point $p \in N$ and it is always possible to introduce local coordinates $X^i = (X^A, X^\alpha)$, $A = 1, \dots, k$, $\alpha = 1, \dots, s = n - k$ in N such that S can be described by $X^A = \text{const}$. dX^A span the k -dimensional kernel of $P(p)$ then and the restriction of P onto S , $P|_S$, is nondegenerate. Since the restriction of a Poisson bracket to functions on a submanifold yields again a Poisson bracket, the inverse of $P|_S$ is a symplectic (i.e. closed and nondegenerate) two-form $\Omega \in \Lambda^2 T^*S$. By means of an appropriate change of variables X^α , it is now always possible to locally bring Ω into Darboux form (cf., e.g., [27, 28]) simultaneously on any of the symplectic leaves S .

In the generic case dX^A spans the kernel of P in a neighborhood of p so that locally the Poisson tensor takes the form: $P = \sum_{l=1}^{s/2} \frac{\partial}{\partial q^l} \wedge \frac{\partial}{\partial p_l}$. (Of course s is an even integer as $\det \Omega \equiv \det \Omega^T = (-1)^s \det \Omega \neq 0$.) In the following we shall call any coordinate system in which P takes this simple form a Casimir-Darboux coordinate system $(X^A, X^I) \equiv (X^A, q, p)$. The case of a singular point is included [29], [25], if one adds to the previous expression for P the term $U = (1/2) \sum_{u,v=1}^{n-s} U^{uv} \frac{\partial}{\partial X^u} \wedge \frac{\partial}{\partial X^v}$, where U depends only on the coordinates X^u ; U is a Poisson structure by itself and vanishes at the considered point p .

Vice versa, it is obvious that *any* choice of a (generalized) foliation of a

manifold N into symplectic leaves, such that the symplectic two-form Ω on each of them can be extended into a smooth two-form $\tilde{\Omega}$ on N , defines a Poisson structure P on N .

If there is an additional structure defined on N , giving rise to a referred coordinate system, Poisson structures identified in the above considerations may need to be distinguished. For instance the manifold N could be a linear space such that only linear transformations on X are admissible. We then find that, the choice of a Poisson structure P linear in these coordinates, $P^{ij} = f^{ij}_k X^k$, is equivalent to the specification of a Lie algebra (since (2.3) reduces to the Jacobi identity for the coefficients f^{ij}_k), whereas a polynomial P yields a W -algebra. Another instance where some coordinates are distinguished on N is the case where N is some Lie group G . A Poisson structure satisfying some specific compatibility condition with respect to the group multiplication on G is called a Lie Poisson structure, the current interest in which stems from the fact that it provides the classical limit of a quantum group [30]. In the context of the gravity models considered in chapter 3 N will play the role of a target space (cf. also the Introduction). In this case the additional input to the otherwise N -diffeomorphism invariant theory will stem from the *interpretation* of specific coordinates as gravity variables; e.g. within the Katanaev-Volovich model (1.4) the coordinate X^3 will play the role of the curvature scalar on the underlying world-sheet or space-time manifold M , as an indirect consequence of the identification (1.7).

To obtain the most general solution to (2.3) in terms of explicit functions on N , we only need to apply a general diffeomorphism $X \rightarrow Y$ to the 'Casimir-Darboux form' of the Poisson structure obtained above:¹

$$P^{ij}(Y) = \left(\frac{\partial X(Y)}{\partial Y} \right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{s \times s} \end{pmatrix} \left(\frac{\partial X(Y)}{\partial Y} \right)^{-1,T}. \quad (2.4)$$

Any choice of the functions $X(Y)$ will lead to a Poisson tensor $P^{ij}(Y)$. Note that one traded in the complexity of finding all solutions to the Jacobi identity

¹Let us here only be interested in the generic local shape of P , i.e. in P in the vicinity of nonsingular points.

(2.3) in favour to the existence of integral surfaces of Hamiltonian vector fields as well as the closure of the forms $\Omega = P|_{\mathcal{S}}^{-1}$ on these surfaces, incorporated in (2.4) through the existence of Casimir-Darboux coordinates X .

The first $k = n - s$ functions $X^A(Y)$ in (2.4) are a (locally) complete set of independent Casimir functions of P , i.e. of those functions which have vanishing Poisson bracket with any other function on N , or, equivalently, which are invariant under the flow of any Hamiltonian vector field $P(dF, \cdot) \equiv \{F, \cdot\}$.

Not any choice of the remaining s functions leads to different functions $P^{ij}(Y)$. To not end up with an overcomplete parametrization of P , we have to factor out the canonical transformations on the symplectic leaves $X^A = \text{const}$. This can be most easily done by requiring that one of the functions $X^I(Y)$ shall be the identity map. The attainability of this gauge, e.g. in the form $X^n(Y) = Y^n$, can be seen by performing the diffeomorphism leading to (2.4) within two steps: Let the original coordinates in which P has Casimir-Darboux form be \tilde{X} . Firstly we perform a canonical transformation such that the n -th new coordinate X^n becomes an arbitrarily prescribed function of the old coordinates \tilde{X}^i ; this is always possible as is seen by inspection of the (infinitesimal) action of a Hamiltonian vector field on a coordinate function, given that P is in Casimir-Darboux form. After this we perform a second diffeomorphism $Y = Y(X)$ which is the identity map in the last component ($Y^n = X^n$). Both steps together clearly provide a completely general coordinate transformation $\tilde{X} \rightarrow Y$. However, the first of them does not change the form of P .

Equation (2.4) is a general local solution, but has the disadvantage that it involves the inverse of matrices. For practical purposes its applicability might therefore be restricted to lower dimensions. In the case that P has at most rank two, there is an alternative form for P which avoids taking the inverse [31]. To derive it, let us first rewrite the expression for the corresponding Poisson brackets in its Casimir-Darboux form:

$$\{F, G\} = \frac{\partial F}{\partial X^{n-1}} \frac{\partial G}{\partial X^n} - (F \leftrightarrow G) = dX^1 \wedge \dots \wedge dX^{n-2} \wedge dF \wedge dG / d^n X.$$

Under a diffeomorphism the volume element $d^n X$ changes only by a multi-

plicative function so that after a general coordinate transformation we find:

$$\{F, G\} = f dC^1 \wedge \dots \wedge dC^{n-2} \wedge dF \wedge dG / d^n X, \quad (2.5)$$

where f and C^1, \dots, C^{n-2} are arbitrary functions, the latter $n - 2$ ones being obviously the Casimir functions.

There still is a further reason for the interest in explicit formulas such as (2.4) and (2.5). Although, if no coordinate system is distinguished in N , all Poisson tensors obtained from the formulas correspond to the same Poisson structure locally, they can be different from a *global* point of view. This happens precisely, when the foliations of N into symplectic leaves are topologically different. The symplectic leaves are (at generic points) the level surfaces of the Casimir functions. So, by appropriately choosing the Casimir functions within (2.4) and (2.5), one can systematically construct Poisson structures, not related to each other by a 'Poisson diffeomorphism' [25].

Let us, as an application of the above formulas, find the most general Poisson structure in a three-dimensional space N which is rotation invariant with respect to the X^3 -axis. From (2.5) we learn

$$P^{ij} = \varepsilon(ijk) f C_{,k}, \quad (2.6)$$

where $\varepsilon(ijk)$ is the alternating symbol. The latter is already invariant under rotations (connected to the identity). Now, C needs to be $SO(2)$ -invariant, since its level surfaces are the integral surfaces of P . Thus also f has to be invariant. So the most general P which is invariant under $SO(2)$ resp. $SO(1, 1)$ transformations in any $(X^3 = \text{const})$ -plane $\subset \mathbb{R}^3$ is provided by Eq. (2.6), in which the free functions f and C depend only on X^3 and $(X)^2 := (X^1)^2 \pm (X^2)^2$.

We noted already that for a linear P , $P^{ij} = f^{ij}_k X^k$, the coefficients f^{ij}_k are structure constants of some Lie algebra \mathfrak{g} . Obviously in this case the vector fields $V^i = \{X^i, \cdot\}$ generate (co)adjoint transformations on the space $N = \mathfrak{g}^*$. The V^i form an overcomplete basis in TN and the symplectic leaves coincide with the coadjoint orbits in \mathfrak{g}^* . The corresponding symplectic

form, introduced and studied by Kirillov [32], Kostant and Souriau [33], is determined through

$$\Omega(V^i, V^j) = \{X^i, X^j\} = f^{ij}_k X^k, \quad (2.7)$$

and plays some role in the representation theory of Lie groups [34].

Before closing this section, let us consider the case that one wants to quantize some symplectic manifold (S, Ω) , not necessarily diffeomorphic to \mathbb{R}^s ; it could be any symplectic leaf of a given Poisson structure P in N . Within the framework of geometric quantization [28] the wave functions are sections in a Hermitian line bundle over S with curvature Ω/\hbar . Such a line bundle exists, iff Ω is 'integral', i.e. iff

$$\int_{\sigma} \Omega = 2\pi n \hbar \equiv nh, \quad n \in \mathbb{Z}. \quad (2.8)$$

for any two-surface $\sigma \subset S$. This is a consequence of the fact that in a line bundle the parallel transport with respect to ∇ around a closed curve γ , $\exp(i \oint_{\gamma} \Theta/\hbar)$, where (locally) $\Theta = d^{-1}\Omega$, can be equivalently expressed as $\exp(i \oint_{\Sigma} \Omega/\hbar)$, if $\partial\Sigma = \gamma$. Different choices of Σ then yield the necessity of (2.8). Since Ω is closed, this condition is empty, if the second fundamental group of S , $\Pi_2(S)$, is trivial. If $\Pi_1(S)$ is trivial, furthermore, the line bundle is unique; otherwise there arises some arbitrariness in the quantization, which can be parametrized by the irreducible representations of this $\Pi_1(S)$ [28], [35].

As an example let us regard the coadjoint orbits S of $N = so(3)^*$. The coadjoint transformations are rotations about the origin and thus the symplectic leaves S are two-spheres characterized by their radius $r = \sqrt{X^i X^i}$. Only for $r = 0$ the symplectic leaf shrinks to a point. To evaluate (2.8) we first have to determine Ω . This can be done most easily by noting that Ω has to be rotation invariant (as rotations are generated by Hamiltonian vector-fields) and that it has to be linear in r , cf. Eq. (2.7). Thus one finds

$$\Omega = r \sin \vartheta d\vartheta \wedge d\varphi = dX^3 \wedge d\varphi, \quad (2.9)$$

where r , ϑ , and φ are the standard spherical coordinates. X^3 and φ are seen to be possible Darboux coordinates for Ω . Here Ω can be rewritten as

$\varepsilon_{ijk}X^i dX^j \wedge dX^k / X^i X_i$, but, although suggestive, such an explicit formula for Ω in terms of structure constants does not exist for general coadjoint orbits. Combining (2.8) with (2.9) we find that only the spheres of radius $n\hbar/2$, $n \in N_0$ are quantizable symplectic manifolds.

For $r = n\hbar/2$ the space of sections in the line bundle over S is infinite dimensional; it is the result of prequantization. The final quantum theory on a (quantizable) symplectic leaf (S, Ω) is obtained after choosing a polarization, which is implemented as a horizontality condition on the sections; for a holomorphic polarization this reads $\nabla_{\bar{z}}\psi(z, \bar{z}) = 0$. The dimension of the resulting Hilbert space is $n + 1$; it is an irreducible representation of $SU(2)$ for spin $n/2$.

Note that if instead of geometric quantization we applied a rather algebraic approach to quantize the Poisson bracket relations $\{X^i, X^j\} = \varepsilon(ijk)X^k$, ignoring for a moment the constraint $X^i X^i = r^2 = \text{const}$, we get the quantization condition $r^2 = n(n + 1)/4$ instead. This difference is at the heart of the ongoing discussions on the different spectra obtained when quantizing $2D$ Yang-Mills theory [36].

The quantization condition (2.8) can be also obtained from a path integral point of view [37]. The action for the above $so(3)$ -invariant point particle systems is of a Wess-Zumino type, which we want to write as $L_p = \int \Theta$ where $\Theta = \Omega$ denotes the canonical potential of the Kirillov form Ω . However, there is no globally well-defined Θ as $\oint_S \Omega \neq 0$ (for $r \neq 0$). Being interested in the path integral, we thus content ourselves with an action L_p that is well-defined up to the addition of a multiple of 2π . The most general $\Theta = d^{-1}\Omega$ on S^2 without poles has the form $r \cos \vartheta d\varphi + \lambda d\varphi$. The above condition on the action then leads to $(r \pm \lambda)\hbar \in Z$ (point particle trajectories close to the poles) which again yields $r = n\hbar/2$. Let me remark here in view of (1.9) that as an action for a two-dimensional field theory $\int \Omega$ obviously *is* well-defined globally also for $S = S^2$.

2.2 A General Action in Two Dimensions

As will be shown in this section, any Poisson structure P on a manifold N induces canonically a topological field theory on a given two-dimensional world sheet manifold M . By means of an additional volume form ε on M and a Casimir function $C^1(X)$ of P one can add to this, furthermore, a nontrivial Hamiltonian.

On the bundle $E \equiv \Lambda^{1,1}(T^*(M \times N))$ there is a canonical form A . In local coordinates x and X of M and N , respectively, it can be written as $A = A_{\mu i} dx^\mu \wedge dX^i$. Let Φ be the map from the world sheet manifold M to E ; if Π_M denotes the projection in the bundle E on M , then let $\Pi_M \circ \Phi$ be the identity map. The topological part of the action we postulate has the following form

$$L_{top} = \int \Phi^* \left(A + \frac{1}{2} i_A i_A P \right), \quad (2.10)$$

where i_A denotes the insertion and Φ^* is the pull back of the map Φ . L_{top} is manifestly invariant under separate diffeomorphisms on M and N . Note, however, that it is not invariant under (arbitrary) diffeomorphisms on $M \times N$ due to the special form of the fiber E . Since L_{top} is (M) -diffeomorphism invariant without the use of a (background) metric on M , and since this feature holds also on the quantum level, it is 'topological' [13]; below it will be found to be of the Schwarz type.

With the additional input of a ('background') volume form ε and the choice of a Casimir function C^1 , we can extend this action with

$$L_+ = \int \varepsilon \Phi^* C^1. \quad (2.11)$$

In coordinates the action $L = L_{top} + L_+$ takes the form

$$L = \int_M d^2x \left\{ \varepsilon(\mu\nu) [A_{\mu i}(x) X^i_{,\nu}(x) + \frac{1}{2} P^{ij}(X(x)) A_{\mu i}(x) A_{\nu j}(x)] + \tilde{\varepsilon}(x) C^1(X(x)) \right\}, \quad (2.12)$$

where $\varepsilon(\mu\nu)$ denotes the alternating symbol and we defined $\tilde{\varepsilon}$ according to $\varepsilon(x) = \tilde{\varepsilon}(x) d^2x$. Displaying only the X coordinates, S becomes

$$L = \int_M A_i \wedge dX^i + \frac{1}{2} P^{ij}(X) A_i \wedge A_j + \varepsilon C^1, \quad (2.13)$$

where we suppressed writing the pull-back.

Let us now turn to the question of local symmetries starting with the simplest situation $P \equiv 0 \equiv C^1$. In this case $L = \int \Phi^* A$ and the most general symmetry of the action up to a total divergence, i.e. up to a local exact term, is provided by $A \rightarrow A + d\epsilon$.

The other extreme case is that P is invertible. In this case we first try to lift the diffeomorphisms in the N space. We noted already that L is N -diffeomorphism invariant. However, when regarding symmetries of the action as functional of the fields X^i and $A_{\mu i}$, P (and also C^1) are not allowed to transform. Thus we are left only with the symplectomorphisms of P , i.e. those transformations of X whose Lie derivative on P vanishes.² The latter are transformations generated by (locally) Hamiltonian vector fields and they clearly leave A (by appropriate transformations of its components) as well as C^1 invariant. Since an M -dependent diffeomorphism in N does not respect the one-one splitting of the form A , these transformations do not directly transfer to symmetries of L . Rather, under a transformation generated by $\epsilon_i(x)P^{ij}(X)\partial/\partial X^j$ the form A on E picks up also a $\Lambda^{2,0}T^*(M \times N)$ part $i_A i_{d\epsilon} P$. However, shifting A further by $d\epsilon$ the action (2.10) becomes obviously invariant up to a total divergence (note that the insertion of $i_A i_{d\epsilon} P$ into P is zero).

Choosing Casimir-Darboux coordinates as a (local) parametrization of L , one immediately finds the general situation to be a direct superposition of the two cases studied above. Thus, in a somewhat formal manner, one can write for the symmetries of (2.10):

$$\begin{aligned}\delta_\epsilon X^i &= \{\epsilon_j X^j, X^i\}_N = \epsilon_j P^{ji} \equiv i_{dX^i} i_{d\epsilon} P \\ \delta_\epsilon A &= d\epsilon + i_A i_{d\epsilon} P,\end{aligned}\tag{2.14}$$

where $\epsilon = \epsilon_i(x)dX^i$. (We used the suffix N for the Poisson brackets on this space, so as to not get confused with the Poisson brackets of the *field* theory in

²All other N -diffeomorphisms, where also P and C^1 are transformed appropriately, correspond to a different parametrization of one and the same action; after such a change of coordinates it is a different functional of, e.g., $X^1(x)$.

its Hamiltonian formulation introduced below). Under these transformations the action changes by $\int_M \Phi^* d\epsilon$. The appearance of the two-zero form on the righthand side of the second equation (2.14) is somewhat ugly. In the useful halfway component notation of (2.13) the symmetries (2.14) can be rewritten as

$$\delta_\epsilon X^i(x) = \epsilon_j(x) P^{ji}(X(x)) \quad (2.15)$$

$$\delta_\epsilon A_i = d\epsilon_i(x) + P^{lm}{}_{,i} A_l \epsilon_m. \quad (2.16)$$

Variation of (2.13) leads to the field equations

$$dX^i + P^{ij} A_j = 0 \quad (2.17)$$

$$dA_i + \frac{1}{2} P^{lm}{}_{,i} A_l \wedge A_m + \varepsilon(C^1)_{,i} = 0. \quad (2.18)$$

A simple comparison of these first order differential equations with the symmetries above establishes that there will be no local degrees of freedom. (Eq. (2.15) contains k independent local symmetries, if k denotes the number of Casimirs, and (2.16) contains $n - k$ further ones). Thus there will be only a finite number of degrees of freedom and these will be of some global nature.

From these considerations we also see that there are no further local symmetries of our action. In particular, the diffeomorphism invariance of (2.13) for $C^1 \equiv 0$ has to be incorporated already within the symmetries (2.14). Indeed, for any given vector field $\xi = \xi^\mu(x) \partial / \partial x^\mu$ generating diffeomorphisms on the world sheet manifold M , the (field dependent) choice $\epsilon := i_\xi A$ in (2.15, 2.16) results in

$$\begin{aligned} \delta_{i_\xi A} X^i &\equiv \mathcal{L}_\xi X^i - i_\xi (dX^i + P^{ij} A_j) \\ \delta_{i_\xi A} A_i &\equiv \mathcal{L}_\xi A_i - i_\xi (dA_i + \frac{1}{2} P^{lm}{}_{,i} A_l \wedge A_m), \end{aligned} \quad (2.19)$$

where \mathcal{L}_ξ denotes the Lie derivative along ξ . Obviously the additional terms on the righthand side of (2.19) vanish for any solution to the field equations (2.17, 2.18) exactly for $C^1 \equiv 0$.

The action L is in first order form, i.e. it is already a Hamiltonian action. A Hamiltonian formulation for infinite dimensional systems requires

appropriate boundary conditions. Therefore we choose M to be of the form $S^1 \times R$, at least locally. More general topologies of M might then be obtained from an appropriate sewing procedure (cf. [38]). We parametrize M by a 2π periodic coordinate x^1 and the 'evolution' parameter x^0 . Note that this does in no way restrict x^0 to be 'timelike' (with respect to whatsoever a metric). As seen most directly from (2.12), A_{1i} is the conjugate variable to X^i , i.e., with the convention $\varepsilon(01) = 1$,

$$\{X^i(x^1), A_{1j}(y^1)\} = -\delta_j^i \delta(x^1 - y^1), \quad (2.20)$$

and the Hamiltonian is

$$H = \int dx^1 (\tilde{\varepsilon} C^1 - A_{0i} G^i) \quad (2.21)$$

with $(\partial := \partial/\partial x^1)$

$$G^i \equiv \partial X^i + P^{ij} A_{1j} \approx 0. \quad (2.22)$$

The weak equality sign ' \approx ' indicates that the $G^i(x^1)$ are zero only on-shell [39], as enforced by means of the Lagrange multipliers A_{0i} , which we may regard as arbitrary external functions on the phase space. By means of the Poisson brackets (2.20) the constraints (2.22) can be easily seen to generate the (one-components) of the symmetry transformations (2.15, 2.16). They are first class constraints, i.e. they close on-shell with respect to the Poisson bracket (2.20):

$$\{G^i, G^j\} = P^{ij}{}_{,k} G^k \delta, \quad (2.23)$$

where we suppressed writing arguments. Let me remark that in order to have the first class property for (2.22) the Jacobi identity for P^{ij} , Eq. (2.3), is not only sufficient but also necessary (whereas for $n \equiv \dim N > 3$, the requirement to have the Hamiltonian vector fields of P on N to be in involution does not lead to (2.3)). It is now straightforward to check that the x^0 -evolution generated by H reproduces (2.18) as well as the zero components of (2.17) (whereas the one-components of Eq. (2.17) are identical to (2.22)). Most of the gauge freedom has been separated to the freedom in choosing A_{0i} (axial gauge) and the constraints G^i generate the corresponding residual gauge freedom.

There is also another way to interpret the symmetries generated by the first class constraints: Allowing for x^0 -dependent coefficients, the G^i can generate all symmetries (2.15, 2.16) for X^i and A_{1i} ; the transformation of the Lagrange multiplier fields A_{0i} can then be determined — in a closed form for a general constraint algebra — by requiring that the Hamiltonian action shall be invariant up to a total derivative [26]. The net result then coincides with (2.15, 2.16).

Let us briefly discuss the BRS formulation of the model (2.10). The main idea of the BRS technique is to enlarge the phase space by introduction of (in our case) fermionic degrees of freedom ('ghosts'), destroying the local symmetry of the action, or rather turning it into a global one generated by the nilpotent BRS charge Q . The 'physical' content of the theory, i.e. the reduced phase space, is reobtained, when one passes to the cohomology of Q . There are some advantages of the Hamiltonian BRS formulation against the Lagrangian one: the BRS Q does not depend on the chosen gauge, the BRS transformations are canonical transformations in the enlarged phase space, and there is the canonical symplectic measure for the definition of the path integral. Note that also covariant gauge conditions may be introduced in the Hamiltonian formalism; one only has to trivially enlarge the phase space by including the Lagrange multiplier fields A_{0i} together with momenta for them which are constrained to zero [26].

In our case the BRS charge is extremely simple. Despite the appearance of structure functions in the constraint algebra (2.23), it still has the minimal form

$$Q = \eta_i G^i - \frac{1}{2} \eta_i \eta_j P^{ji}{}_{,k} \mathcal{P}^k, \quad (2.24)$$

where (η_i, \mathcal{P}^i) are canonically conjugate fermionic ghost variables associated with $G^i \approx 0$. It is straightforward to verify $\{Q, Q\} = 0$ as a consequence of (2.3). Note that, depending on the chosen gauge fermion K , there still may appear quartic ghost vertex contribution in the quantum action; as, e.g., it may happen in the multiplier gauge $K = -\chi_i \mathcal{P}^i$, if the gauge conditions $\chi_i(A_{1i}, X^i)$ do not commute with $P^{ji}{}_{,k}$.

It would be desirable to extend the formulation of (2.10) so as to ex-

plicitely include the case of nontrivial fiber bundles over M . A can then be a one-form only locally, i.e. only on some $U \times N$, $U \subset M$, because one should allow for a nontrivial fibration of N over M . We do not have a satisfactory answer for this problem yet. Probably there should be some superior formulation of the model, maybe including further fields which can be gauged out only locally, or, as in the case of the WZW-theory, having some three-dimensional closed term in its action. Then this other action should be strictly invariant under the symmetry transformations and also the transformation (2.14) for A should be replaced by something more perspicuous.

Some remarks might be in place here:

Firstly, the field equations (2.17, 2.18) are (strictly) covariant under the transformations (2.15, 2.16) as they stem from a local variation of the action. It is no problem to explicitly construct nontrivial bundles over M by means of their solutions, at least if one can integrate the infinitesimal form of the symmetry transformations (2.15, 2.16).

Secondly, as a Hamiltonian system, i.e. on the level of the field equations, the completely gauged WZW theory with compact gauge group G , defined on the cylinder via a Gauss decomposition, turns out to be a special case of our theories with $N = G$ [40]. A natural question then arises: What is the ungauged or partially gauged version of the general model (2.10)?

Thirdly, in the case that we choose a linear Poisson structure, $P^{ij} = f_k^{ij} X^k$, we regain two-dimensional nonabelian gauge theories: In this linear case the N -coordinate X^i is an equally well-behaved object as dX^i ; A_i is then commonly spanned on Lie algebra generators T^i , $A = A_i T^i$, satisfying $[T^i, T^j] = f_k^{ij} T^k$, and similarly X^i is spanned on a dual basis. After addition of $\int d(A_i X^i)$ to (2.13) and a partial integration, L_{top} takes the standard form

$$L_{top} = \int_M X^i F_i, \quad (2.25)$$

where $F = dA + A \wedge A$. The standard 2D Yang-Mills theory $\int tr(F \wedge *F)$ for semisimple groups is obtained, in its first order form, when choosing $C^1 \propto tr(XX)$ and setting ε equal to the metric induced volume form used to define the Hodge dual $*$. Thus in this linear case, and only there, the

addition of the surface term $\int d(A_i X^i)$ provides an action which is strictly invariant under the transformations (2.15, 2.16); and certainly here we also know how to (simply) understand nontrivial fiber bundles.

Last but not least, it might be interesting to study the limit $n \rightarrow \infty$ of the dimension of target space N of the model (2.10) [41]. In this way one may gain topological field theories in higher dimensions. For instance, an application of such a limit to the completely gauged WZW model was shown to result in fourdimensional selfdual gravity.

(During the completion of this work my colleague P. Schaller found the Wess Zumino type formulation of (2.10). E.g., in the case of the gauge theories (2.25) the corresponding boundary term has the form $\int_B DX^i F_i$, where D denotes the covariant derivative. It obviously reproduces (2.25) due to $DF \equiv 0$, if $M = \partial B$.)

2.3 The Classical Solutions

There are basically two ways to solve the field equations (2.17, 2.18). One can either choose gauge conditions or one can work in appropriate target space coordinates, most referably in Casimir-Darboux coordinates of P (cf. section 2.1). In any case one will have to determine the Casimir functions of P . So to start with let us suppose one knows how to cover N by charts U_a in each of which one has Casimir-Darboux coordinates X^A, X^I of P . By continuity of all maps Φ from M to N , any *global* solution Φ satisfying the field equations can be obtained by an appropriate patching on M .

Let the first Casimir coordinate X^1 coincide with the Hamiltonian C^1 . The independent field equations then take the simple form ($C^1 \neq 0$):

$$dX^A = 0 \tag{2.26}$$

$$dA_1 = -\varepsilon \tag{2.27}$$

$$dA_A = 0, \quad \text{for index } A \neq 1 \tag{2.28}$$

$$A_I = \Omega_{JI} dX^J, \tag{2.29}$$

where taking the pullback is understood implicitly. In the case $C^1 \equiv 0$ one

has $dA_1 = 0$ instead of (2.27). Obviously the general solution to the above equations can be obtained without any choice of a gauge: The Casimir fields $X^A(x)$ have to be constant on M , but otherwise arbitrary, whereas the $X^I(x)$ remain completely undetermined by the equations of motion. Any choice of the latter determines A_I uniquely through (2.29). For $C^1 \equiv 0$, $A_A = df_A$, finally, with arbitrary functions f_A . For $C^1 \not\equiv 0$, there always exists a local solution to (2.27) because ε is closed, and A_1 is again determined only up to an exact one-form df_1 . (E.g. in coordinates on M such that $\varepsilon = d^2x$: $A_1 = x^0 dx^1 + df_1$).

Up to now one has not made use of the gauge freedom. As is obvious from (2.15, 2.16) any choice of the X^I is gauge equivalent, and also $A_A \sim A_A + dh_A$, where the h_A are arbitrary functions. Thus *locally* any solution to the field equations is uniquely determined by the values of the Casimir functions.

Additional structure evolves, if global aspects are taken into account. In a completely coordinate independent manner the field equations (2.17) take the form

$$\Phi^*(e + i_A i_e P) = 0 \quad \forall e \in T^*N, \quad (2.30)$$

where, as before, Φ denotes the map from M into $F = \Lambda^{1,1}T^*(M \times N)$, the projection Φ_M of which onto M is trivial. Reformulating the previous local considerations N -coordinate indepently, we find all global solutions to (2.30):³

1) $\Phi_N \equiv \Pi_N \circ \Phi$, where Π_N denotes the projection in F to N , may be an arbitrary map from M into any symplectic leaf (or integral surface) $S \subset N$ of P . All smooth deformations of this embedding of M into S are gauge transformations.

2) The restriction of A to TS , $A|_S$, (i.e. the pullback of A with respect to the embedding function of S into N), is then uniquely determined by:

$$\Phi^*[i_v(A + \Omega)] = 0 \quad \forall v \in TS, \quad (2.31)$$

where $\Omega \in \Lambda^2 T^*S$ denotes the inverse of $P|_S$.

³In the case of a nontrivial fiber bundle one merely replaces 'map' by 'section'.

The remaining field equations (2.18) may not that easily be rewritten in a completely coordinate independent manner. Still we know that in any local Casimir-Darboux coordinate system on N the only remaining equations to be solved are (2.27, 2.28). Also all the remaining local symmetries can be integrated easily in this coordinate system yielding $A_A \sim A_A + dh_A$. In many cases this will suffice to classify solutions globally; examples for this shall be provided elsewhere.

In some instances it may be favorable to express (2.27, 2.28) in a more general coordinate system. Let the first k coordinates still be Casimir coordinates X^A , but the remaining $s = n - k$ coordinates X^α be arbitrary. Then for $i = A$ Eq. (2.18), with, for simplicity, $C^1 \equiv 0$, takes the form⁴

$$dA_A + \Omega_{\alpha\beta,A} dX^\alpha dX^\beta = 0. \quad (2.32)$$

One can check further that Eq. (2.18) with $i = \alpha$ is already fulfilled by the solutions **1** and **2** above, i.e. that these equations are already a consequence of (2.17). Locally, Eq. (2.32) can be integrated easily: Up to gauge transformations one finds $A_A = \Theta_{\alpha,A} dX^\alpha$, if Θ again denotes a symplectic potential for Ω .

Before turning to the quantum theory, let me briefly comment on the option of solving the field equations by first choosing gauge conditions. Probably the most efficient one is $A_{0i} = 0$. Locally this gauge is always attainable by means of the symmetry transformations (2.15, 2.16), as is most easily seen in the Hamiltonian formulation of the theory, where the A_{0i} are subject to arbitrary shifts.⁵ For $C^1 \equiv 0$, e.g., the field equations state in this gauge that all fields are x^0 -independent and that they have to satisfy the constraint equations (2.22). The general solution to the latter is then provided by **1** and **2** above, where M is replaced by an open interval.

⁴As already in (2.17, 2.18) we suppress writing the pullback in the following.

⁵In the gravity theories considered in the following chapter the gauge $A_{0i} = 0$ corresponds to a degenerate metric, as is obvious from (1.7). One of the simplest choices not in conflict with the metric nondegeneracy is $e_0^- = \omega_0 = 0$, $e_0^+ = 1$, used, e.g., in [22]. The consequences of the fact that the Hamiltonian symmetries connect degenerate with nondegenerate metric configurations is studied in detail in Sec. 3.5.

2.4 All Quantum States

Up to now an exact treatment of standard quantum field theories is beyond human abilities and one takes recourse to approximative methods such as perturbation theory. In our case, however, a nonperturbative treatment is accessible.

The Hamiltonian formulation of our theory has been presented already in section 2.2. It corresponds to a worldsheet topology $M = S^1 \times R$. Let us consider the wave functionals in an X -representation. Any quantum wave function Ψ is then a complex-valued functional of parametrized smooth loops $\mathcal{X} : S^1 \rightarrow N$ in N . However, following Dirac [39], only such quantum states are admissible which satisfy the quantum constraints

$$\hat{G}^i(x)\Psi[\mathcal{X}] = \left(\partial X^i(x) + i\hbar P^{ij}(X) \frac{\delta}{\delta X^j(x)} \right) \Psi[\mathcal{X}] = 0, \quad (2.33)$$

resulting from (2.22) by the replacement $A_{1i}(x^1) \rightarrow i\hbar\delta/\delta X^i(x^1)$, suppressing the superscript one for the variable x^1 within this section. It is decisive that the operator ordering within the $\hat{G}^i(x)$ is such that taking commutators between the quantum constraints does not produce further constraints. (2.33) still leads to the constraint algebra (2.23).

Maybe less well-known is an additional restriction on the operator ordering within G , probably present in an analogous manner in most diffeomorphism invariant theories. Any gauge invariant phase space function, such as, e.g., the Casimir functions $C^A(X(x))$, have to be x -independent on-shell because diffeomorphisms are part of the symmetry transformations (cf. also Eqs. (2.19)). Therefore $\partial C^A(X(x))$ will be part of the field equations, and, as such, it will result from an appropriate combination of the constraints. Since, however, $\oint \partial C^A dx \equiv 0$, the integral over the corresponding combination of the constraints will also vanish identically; in our case

$$0 \equiv \oint \partial C^A dx^1 \equiv \oint \frac{dC^A}{dX^i} G_i. \quad (2.34)$$

This indicates a subtle dependence of the constraints among each other. It is decisive to maintain the relations (2.34) also on the quantum level, and we

have done so in (2.33). This is maybe best illustrated at the simple example of two classical constraints $g^1 = g^2 = qp$, in a phase space of some arbitrary dimension, which obviously satisfy $g^1 - g^2 = 0$; The operator ordering $\hat{g}^1 = -i\hbar q(d/dq)$, $\hat{g}^2 = -i\hbar(d/dq)q$ does not produce any anomaly in the (trivial) constraint algebra, but obviously there is *no* nontrivial common kernel of the constraint operators as $\hat{g}^1 - \hat{g}^2 = -i\hbar$.

Again the solution to (2.33) is most easily found in Casimir coordinates (X^A, X^α) of P . In a parallel way as we obtained solution **1** of the previous section we find that the support of the wave functionals has to be on such loops which lie entirely within some integral surface S of P . Next we have to solve

$$\left[\frac{\delta}{\delta X^\alpha(x)} - \frac{i}{\hbar} \Omega_{\alpha\beta}(X(x)) \partial X^\beta(x) \right] \Psi[\mathcal{X}] = 0. \quad (2.35)$$

This equation can be reinterpreted as a *horizontal condition* for the complex-valued functionals on the space Γ_S of loops on each S (but not on N , except for an invertible P resulting in $S = N$):

$$(d + (i/\hbar)\mathcal{A})\Psi = 0, \quad (2.36)$$

where the $U(1)$ -connection is uniquely defined via

$$\mathcal{A}\left(\frac{\delta}{\delta X^\alpha(x)}\right) = -\Omega_{\alpha\beta}(X(x)) \partial X^\beta(x). \quad (2.37)$$

A necessary condition for (2.36) to have nontrivial solutions is that \mathcal{A} is closed. According to (2.37) there is a close relationship between \mathcal{A} , which is a connection in a $U(1)$ -bundle over the loop space Γ_S , and the symplectic form Ω on the underlying space S . Let us make this relationship more precise, so that finally $d\mathcal{A} = 0$ will be a simple consequence of $d\Omega = 0$.

Any two-form ω on a manifold S generates a one-form α on the loop space Γ_S on S : Forms are basically the dual objects to areas of integration. Now, any path γ in Γ_S , corresponding to a one-parameter family of loops in S , obviously spans a two-dimensional surface $\sigma(\gamma)$. Thus, given $\omega \in \Lambda^2 T^*S$, we can uniquely define α via

$$\int_\gamma \alpha = \int_{\sigma(\gamma)} \omega; \quad (2.38)$$

in this equation we assigned a number to any path $\gamma \in \Gamma_S$, which, by duality, defines the one-form α . Since, furthermore, any closed path γ corresponds to a closed two-surface $\sigma(\gamma)$, α is closed, iff ω is closed.

Of course, not every one form on Γ_S can be described in this way. In our case, however, the one form \mathcal{A} on Γ_S , is indeed generated by Ω . To prove this let us choose a path $\gamma \in \Gamma_S$ parametrized by a parameter $\tau \in [0, 1]$. Any point in γ corresponds to a loop \mathcal{X} . Thus γ induces a map $S^1 \times [0, 1] \rightarrow N : (x, \tau) \rightarrow X(x, \tau)$, which precisely corresponds to a parametrization of $\sigma(\gamma)$ introduced above. Denote by $\dot{\mathcal{X}} \in T\Gamma_S$ the tangent vector to γ : $\dot{\mathcal{X}} = \oint dx \dot{X}^\alpha(x, \tau)(\delta/\delta X^\alpha(x))$, where the dot denotes the derivative with respect to τ . Then, as an obvious consequence of (2.37),

$$\begin{aligned} \int_\gamma \mathcal{A} &= \int_0^1 \mathcal{A}(\dot{\mathcal{X}}) d\tau = \\ &= - \int_0^1 \oint_0^{2\pi} \dot{X}^\alpha(x, \tau) \Omega_{\alpha\beta}(X(x, \tau)) \partial X^\beta(x, \tau) dx d\tau = \int_{\sigma(\gamma)} \Omega. \end{aligned} \quad (2.39)$$

Thus (2.33) can be integrated locally to yield $\Psi = \exp[(i/\hbar) \int \mathcal{A}] \Psi_0$ for any initial value Ψ_0 . The integrability extends to a global one, if Γ_S is simply connected, i.e. if $\Pi_2(S)$ is trivial. For the case that $\Pi_1(\Gamma_S) \equiv \Pi_2(S) \neq 1$, however, the one-valuedness of a nontrivial Ψ is given, if and only if \mathcal{A} is integral, i.e. iff

$$\oint_\gamma \mathcal{A} = nh, \quad n \in Z \quad (2.40)$$

for any (noncontractible) closed loop γ representing an element of $\Pi_1(\Gamma_S)$. Due to (2.39) this is equivalent to an integrality condition for Ω :

$$\oint_\sigma \Omega = nh, \quad n \in Z \quad (2.41)$$

for any (noncontractible) closed two-surface σ representing an element of $\Pi_2(S)$.

Let us denote the space of symplectic leaves by \mathcal{S} . As this is the space N modulo the flow of the Hamiltonian vector fields, \mathcal{S} in general is no more a smooth manifold. Nevertheless, at least when ignoring the symplectic leaves of less than maximal dimension s , any atlas (X^A, X^α) , where again the X^A are local Casimir coordinates, can be used to define an atlas of \mathcal{S} with local

coordinates X^A . ($X^A = \text{const}$ characterizes then all of the orbit, even if it leaves the chart (X^A, X^α)). Now, if some symplectic leaves $S \in \mathcal{S}$ have a nontrivial second fundamental group, the integrality condition (2.41) may yield a restriction of the support of Ψ to loops on a (possibly discrete) subset of \mathcal{S} , which we name $\tilde{\mathcal{S}}$. For $S \in \tilde{\mathcal{S}}$, Ψ is determined up to a multiplicative constant on any connected component of Γ_S . As the space of connected components of Γ_S is in one to one correspondence with the first homotopy group of S , we may identify physical states with complex valued functions on \mathcal{I} defined via

$$\mathcal{I} = \bigcup_{S \in \tilde{\mathcal{S}}} \Pi_1(S) \quad \tilde{\mathcal{S}} = \{S \in \mathcal{S} : \Omega \text{ integral}\} . \quad (2.42)$$

If $\Pi_2(S) = 1 \forall S \in \mathcal{S}$, then $\tilde{\mathcal{S}} = \mathcal{S}$. Using the atlas for (the generic parts of) \mathcal{S} introduced above in this case, in local coordinates the physical wave functions take the form

$$\Psi[\mathcal{X}] = \Psi_0(X^A, n_A) \exp \left[(i/\hbar) \int_D \Omega \right] \quad , \quad \partial D = \text{Image} \mathcal{X} . \quad (2.43)$$

Here n_A denotes a discrete index labelling the elements of $\Pi_1(S)$ and, as a consequence of (2.41), the choice of the disk $D \subset S$, whose boundary is the considered loop \mathcal{X} of the functional, is arbitrary. In the case that $\Pi_2(S)$ is nontrivial for some $S \in \mathcal{S}$, the wave functions may again be represented by (2.43) where, however, the range of the coordinates X^A has to be restricted such that (2.41) is fulfilled; certainly this can have the effect of partially replacing some of these coordinates by discrete labels.

Note that the phase in (2.43), which is not invariant under general classical P -morphisms, basically coincides with the action of the point particle system on the symplectic leaf studied in Sec. 2.1. The reason for obtaining the same integrality condition (2.33) can be understood by observing that in any case we want the phase factor to be well-defined; in the case of the point particle system, so as to have a well defined path integral, and in the case of the topological field theory, so as to obtain a smooth wave-functional. Note also the difference between the two quantum theories: The point particle system is defined inherently on a symplectic leaf and, e.g., for an $su(2)^*$ -orbit

of radius $n\hbar/2$, the Hilbert space is of dimension $n + 1$. The wave functions of the field theory, on the other hand, basically reduce to functions on the space $\tilde{\mathcal{S}}$ of (quantizable) symplectic leaves; for $N = su(2)^*$, the number of independent states on any of the (quantizable) symplectic leaves $S \sim S^2$ is just one in this case, since $\Pi_1(S^2) = 1$.

The Hamiltonian (2.21), being constant on each of the symplectic leaves, defines a function on \mathcal{I} and thus becomes a multiplicative operator upon quantization. Considerations on constructing a measure on \mathcal{I} shall be taken up elsewhere, or will be discussed for some of the models considered below (cf. Sec. 3.1).

In the example of nonabelian gauge theories (cf. also [42]) N is the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} of the gauge group. The symplectic leaves are the coadjoint orbits equipped with the Kirillov symplectic form Ω . The integrality condition (2.33) selects those symplectic leaves [32] which are characterized, in the case of a compact semisimple \mathfrak{g} , by values of the Casimir constants lying in the weight lattice; the quantization of these symplectic leaves yields the unitary irreducible representations of \mathfrak{g} . This observation establishes a connection between our representation and the A_{1i} -connection representation of quantum mechanics for nonabelian gauge theories on a cylinder [43]. Details, including a comparison of the spectra (cf. also [36]), shall be provided elsewhere.

In the noncompact case of, e.g., $\mathfrak{g} = so(2, 1)$, all the level surfaces of the Casimir $C = (X^1)^2 - (X^2)^2 + (X^3)^2$ have trivial Π_2 . Thus (2.33) is empty in this case. For $C > 0$ we have $\Pi_1 \sim Z$, furthermore. To display the wave functions we again choose an atlas in N (inducing an atlas in \mathcal{S}): Let us choose (X^1, X^3, C) for $X^2 > 0$ as one of our charts ($C = \text{const} < 0$ corresponds to two symplectic leaves, which may be distinguished by the sign of X^2). We then have $\Psi_0 = \Psi_0(C, n)$, where $n \in \mathbb{N}$, and $\Psi_0(C < 0, n \neq 1) = 0$ since $\Pi_1(S_{C<0})$ is trivial. If one also wants to display $\int_D \Omega$ explicitly, one needs further charts on S ($\Omega_{\alpha\beta}$ might not be well-defined, e.g., in coordinates X^1, X^3 on all of the chart $X^2 > 0$). In the present case of $so(2, 1)^*$ on any of

the symplectic leaves except for the origin Ω can be completely described by

$$\Omega = \pm dX^3 \wedge dX^\pm / X^\pm, \quad (2.44)$$

where

$$X^\pm = \frac{1}{\sqrt{2}}(X^1 \pm X^2), \quad (2.45)$$

whereas $\Omega = 0$ at the singular orbit $X = 0$.

Having found the kernel of the quantum constraints corresponds to dividing out all local symmetries (2.15, 2.16) connected to the identity. Thus the Yang-Mills theories on a cylinder described in this way are the ones for the *universal covering group* of the chosen Lie algebra \mathfrak{g} . The transition to Yang-Mills theories on a cylinder for not simply connected gauge groups calls for further steps; one has to require an appropriate transformation of the wave functions under large gauge transformations, which may further exclude some wave functions. For the case of $SO(2, 1)$ this has been studied in some detail in [10]; the effect is quite drastic: whereas the Yang-Mills Hamiltonian C obviously has a continuous spectrum in the case of $\widetilde{SO}(2, 1)$ (=univ. cov. of $SO(2, 1)$), the spectrum becomes discrete for $C < 0$ in the $SO(2, 1)$ theory.

Chapter 3

Models of Gravity in 1+1 Dimensions

3.1 A Universal Gravity Action and Remarks on its Quantization

Let us construct actions for gravitational theories from the action L introduced in Sec. 2.2. The starting point shall be the identification (1.7), where e_a and ω are the zweibein and spin connection of the gravity theory, respectively. This requires a target space N of a minimal dimension three.

Our conventions concerning the gravity theories shall be summarized as follows: The metric g is obtained through $g = \eta^{ab}e_a \otimes e_b$, where η is the frame metric with signature $(1, \pm 1)$. In the Minkowski case negative lengths shall be interpreted as spacelike distances. Since the structure group of the frame bundle is abelian in two dimensions, the spin connection one-form introduced above has no frame indices and the curvature is just $d\omega$. Contact with formulas used in higher dimensional gravity can be established via $\omega_{ab} = \varepsilon_{ab} \omega$, where ε_{ab} are the covariant components of the ε -tensor $\varepsilon = e^1 \wedge e^2$ ($\Rightarrow \varepsilon_{12} = 1$).¹ So one finds, e.g., that the Hodge dual of the curvature two-

¹Note that in this chapter ε denotes the metric induced volume form and, in contrast to the previous chapter, it is dynamical now.

form, $*d\omega$, equals the half of the Ricci scalar R . For reasons of completeness we will discuss gravity theories for both signatures of the metric in this section, that is Minkowski as well as Euclidean gravity. In the following section, however, we will deal with the Minkowski type theories only. In this case then it will prove useful to introduce light cone coordinates

$$e^\pm = \frac{1}{\sqrt{2}}(e^1 \pm e^2) \quad (3.1)$$

in the frame bundle, which lead to an off-diagonal frame metric $\eta_{+-} = 1$ as well as to $\varepsilon^{+-} = 1$.

The action of a gravity theory has to be invariant against (M) -diffeomorphisms and frame-rotations. The first condition leads to $C^1 \equiv 0$, i.e. to an on-shell vanishing Hamiltonian (cf. Eq. (2.21)). The second condition has been solved in generality in the paragraph of Eq. (2.6). Thus the gravity action we propose has the form of L_{top} , i.e. of Eq. (2.13) with $C^1 \equiv 0$, where the Poisson structure P is defined through (2.6) by specifying f and C as functions of

$$(X)^2 \equiv X^a X_a \equiv (X^1)^2 \pm (X^2)^2 \quad (3.2)$$

and X^3 .

Let us find the subclass of this family of actions which leads to a torsion-free gravity theory. This is equivalent to the search for those actions L with X^a -dependence $\int X^a De_a$, where $De^a \equiv de^a + \varepsilon^a_b \omega e^b$ is the torsion two-form. (One could replace the X^a also by strictly monotonic functions of them, but this does not change the resulting gravity theory). The occurrence of such a term in the Lagrangian requires $f = 1/(2\partial C/\partial(X)^2)$. To have it be the only X^a -dependent term in L , we find that L has to be of the form

$$L = \int_M X_a De^a + X^3 d\omega - V\varepsilon, \quad (3.3)$$

where V is an arbitrary potential of X^3 . This Lagrangian was first proposed in [44].

If we again release the torsion zero condition, V in (3.3) may be chosen as an arbitrary function of X^3 and $(X)^2$ in order to correspond to a rotation invariant Poisson structure; this can be checked by verifying (2.3) directly,

whereas the determination of the functions f and C of (2.6) yielding this Poisson structure appears to be cumbersome.

All of what follows will be based on the action (3.3) with

$$V = v(X^3) + \frac{\alpha}{2}(X)^2, \quad (3.4)$$

where v is some arbitrary function. The second term in this potential allows to add a torsion squared term to the most general torsionless action within our framework (choosing a three dimensional target space). The Poisson structure yielding (3.4) results from the choice

$$C = (X)^2 \exp(\alpha X^3) + 2 \int_0^{X^3} v(y) \exp(\alpha y) dy \quad (3.5)$$

for the Casimir function C and $\exp(-\alpha X^3)/2$ for the integrating factor f . It might be worthwhile to investigate in how far the results found below may be generalized to the case of an arbitrary $((X)^2, X^3)$ -dependence of f and C .

From (3.3, 3.4) one can easily regain other well-known theories of two-dimensional gravity. For instance, integrating out the Lagrange multiplier fields X^a , the choice $V = \Lambda X^3$ yields the Jackiw-Teitelboim model (1.2). The most general quadratic potential $V^{KV} = \mp(X^3)^2 - \Lambda \mp \alpha(X)^2/2$ leads upon elimination of the X -coordinates to the Katanaev-Volovich model (1.4) (use $*\varepsilon = \pm 1$).² In a similar way one obtains the action L^{R^2} for R^2 -gravity (1.3) from the potential $V^{R^2} = (X^3)^2 - \Lambda$. It coincides with (1.4) for Minkowski signature, if the torsion squared term is replaced by a torsion zero condition (limit $\alpha \rightarrow 0$).

Also the model (1.5) can be included: It may be obtained from (3.3) with $V = \Lambda$, if one replaces (1.7) by $A_a = \sqrt{X^3} e_a$ and identifies X^3 with $\exp(-2\Phi)$ [45].³ Promoting Λ to a dynamical field and adding $A_4 d\Lambda$ to the

²Let us remind the reader that the elimination of fields through their *own* equations of motion within an action is always possible, at least on the classical level, as is clear from the variational principle. For the case that the eliminated fields appeared at most quadratic within the original Lagrangian, the same result is obtained when integrating out these fields within the path integral.

³It turns out that it is *always* possible to insure $X^3 > 0$ by means of gauge transformations.

latter action, furthermore, this action can be brought even into the group theoretical form (2.25) for the centrally extended Poincaré group [46]. Allowing for a dynamical constant Λ , one also may obtain a to (1.5) on-shell equivalent action from the (more straightforward) identification (1.7) and $X^3 \sim \Phi$: Choose $V = \exp(\lambda X^3)$ and add $A_4 d\lambda$ to (3.3).

But not only purely two-dimensional models may be incorporated. The choice $V = 1/2(X^3)^2$, e.g., yield solutions which are precisely of the Schwarzschild form, disregarding the rotation invariant part $r^2 d(\cos \theta) d\varphi$; moreover, the Casimir constant may be chosen so as to basically coincide (for one of its signs) with the standard Schwarzschild mass. In this way it is possible to come to a quantum description of the Schwarzschild solution (i.e. of the *spherically symmetric* solutions to the four dimensional Einstein vacuum equations) on topology $S^2 \times S^1 \times \mathbb{R}$.

Let us conclude this section by discussing the quantization of the gravity models (3.3, 3.4). We analyzed already the prototypes $V = \Lambda X^3$ (yielding the action of an $SU(2)$ resp. $\widetilde{SO}(2, 1)$ nonabelian gauge theory). The generalization is quite straightforward. Our first task is to determine the topology of the symplectic leaves, in particular their first and second homotopy groups. As we are in a three dimensional target space N the symplectic leaves S are either two- or zero-dimensional. The latter occurs wherever $P = 0$, i.e. at points

$$X^1 = X^2 = 0, \quad X^3 = B_{crit} = \text{const}, \quad \text{such that } v(B_{crit}) = 0, \quad (3.6)$$

which we will call 'critical' or 'singular' in the following; the corresponding value of the Casimir constant (3.5) is $C_{crit} = 2 \int_0^{B_{crit}} v(y) \exp(\alpha y) dy$. At 'noncritical' values of C the connected components of the level surfaces N_C coincide with the symplectic leaves.

To determine the topology of the level surfaces N_C it is convenient to consider the function

$$\begin{aligned} h(X^3) &:= \exp(2\alpha X^3) (X)^2(X^3, C) \equiv \\ &\equiv \exp(\alpha X^3) [C - 2 \int_0^{X^3} v(y) \exp(\alpha y) dy] \end{aligned} \quad (3.7)$$

obtained from inverting the Casimir function (3.5) for constant C . It is only the number and kind of zeros of h which determines the topology of N_C , where the zeros of h are nonsimple only at critical points $X^3 = B_{crit}$.

For Euclidean signature, resulting from the plus sign in (3.2), one only has to rotate the positive parts of the curve $\exp(-2\alpha X^3) h(X^3)$ around the X^3 -axis to obtain the level surfaces N_C of the Casimir function. Obviously a positive part of h between two successive simple zeros at $X^3 = B_{1,2}$ yields a symplectic leaf isomorphic to a two-sphere. Such a leaf is subject to the integrality condition (2.8), which here takes the simple form

$$\oint \Omega = \int dX^3 \wedge d\varphi = 2\pi[B_2 - B_1] = nh \quad (3.8)$$

as can be seen by expressing P in coordinates (C, φ, X^3) , with $\tan \varphi = (X^2/X^1)$. A positive (part of) h with no (one simple) zero results in a cylindrical (planar) symplectic leaf; the integrality condition is trivial then. Changes of the topology of N_C (along the choice of C) occur only at sliding intersections of h with the X^3 axis, i.e. at critical values of C .

For Minkowski signature the transition from h to N_C is a bit more cumbersome. The result is, however, quite simple: If h contains no points $(X)^2 = 0$, N_C consists of two disconnected 'planes'; if h contains l points of (nonsliding) intersections with the X^3 axis, it has $l - 1$ fundamental noncontractible loops. The second fundamental group is, moreover, always trivial. For the critical values $C = C_{crit}$ (sliding intersections) we again have fixed points of the Hamiltonian vector fields at $(0, 0, B_{crit})$; the set \mathcal{S} of integral surfaces becomes nonHausdorff there, as can be seen already for the $so(2, 1)$ prototype (in the space of coadjoint orbits, which here coincides with the space of Lorentz orbits in \mathbb{R}^3 , the origin and the positive and negative light cones have no disjoint neighborhoods).

Let us discuss R^2 -gravitiy, $V^{R^2} = (X^3)^2 - \Lambda$, to more detail. An analysis of the function $h(X^3) = C + (2/3)(X^3)^3 - 2\Lambda X^3$ yields five qualitatively different cases depending on the parameters C and Λ (see Fig. 1):

- R1** : one single zero of h and thus also of $(X)^2$ at $X^3 = B$
- R2** : one triple zero at 0

R3 : one single zero at B_1 and one double zero at $+\sqrt{\Lambda}$

R4 : one double zero at $-\sqrt{\Lambda}$ and one single zero at B_3

R5 : three single zeros at B_1 , B_2 , and B_3 ,

where $B_1 < -\sqrt{\Lambda} < B_2 < +\sqrt{\Lambda} < B_3$ and $-\infty < B < +\infty$.

Obviously $B_{crit} = \pm\Lambda$ and the curve along **R2,3,4** in Fig. 1 corresponds to the critical values $C_{crit} = C_{<(>)} \equiv \pm(4/3)\Lambda^{(3/2)}$ of C . For $\Lambda < 0$ the quantum theory is particularly simple: For both signatures the symplectic leaves are isomorphic to planes, the spectrum of C is \mathbb{R} , and, up to the phase factor, the wave functions (2.43) are functions of one argument C . For $\Lambda > 0$ and $C \in (-\infty, C_<) \cup (C_>, \infty)$ the resulting surfaces are again manifolds with trivial topology. However, for $C \in (C_<, C_>)$ and Euclidean signature we get two disconnected surfaces of the topology of a plane and a sphere, respectively. Thus the continuous spectrum $C \in \mathbf{R}$ has a twofold degeneracy for some specific values of C within this range $(C_<, C_>)$. For Minkowskian signature and $C \in (C_<, C_>)$ the level surfaces N_C are connected and of trivial second homotopy; however, there are two fundamental noncontractible loops, the winding numbers of which give rise to a quantum number $n_C \in \mathbf{Z}$ within the wave functions (2.43).

Concerning the question of the inner product, let us remark here that on large parts of the phase spaces of any of the models (3.3) with (3.4) and Minkowski signature, the variable conjugate to C can be written as

$$p = -\frac{1}{2} \oint \exp(-\alpha X^3) \frac{e_1^-}{X^-} dx^1 \approx -\frac{1}{2} \oint \exp(-\alpha X^3) \frac{e_1^+}{X^+} dx^1. \quad (3.9)$$

Pulling through the phase factor of (2.43), which in local target space coordinates takes the form

$$\exp\left(-\frac{i}{\hbar} \oint \ln |X^-| \partial X^3 dx^1\right) \sim \exp\left(\frac{i}{\hbar} \oint \ln |X^+| \partial X^3 dx^1\right), \quad (3.10)$$

the Dirac observable p acts via $(\hbar/i)(d/dC)$ on Ψ_0 . Requiring that it will become a hermitian operator severely restricts the measure of the inner product, but, in the case that Ψ_0 depends also nontrivially on quantum numbers n_C and/or that the level surfaces N_C have several disconnected parts, this does not determine the inner product entirely.

In the case of Minkowskian R^2 -gravity with $\Lambda < 0$ there are no such quantum numbers and the measure becomes the ordinary Lebesgue measure in C by the above prescription; thus this theory reduces entirely to the one of an ordinary point particle system on the line.

For Minkowskian R^2 -gravity with $\Lambda > 0$ it is not so clear how to determine the inner product between states of different winding numbers n_C . In this context it seems appropriate to mention that the assignment of winding numbers at the critical values of C is somewhat ambiguous: On the one hand the critical points $(0, 0, \pm\sqrt{\Lambda})$ constitute integral surfaces by themselves and loops in the support of Ψ may not pass these points. On the other hand the critical points and the rest of the orbit(s) at this value of $C = C_{crit}$ do not have disjoint neighborhoods in \mathcal{S} ; so continuous functions Ψ_0 identify them. Let us further note that a Faddeev-Popov kind of prescription for the inner product always will assign measure zero to the loops on singular points; this seems questionable at least in the case $v \equiv 0$, where the Poisson structure vanishes on all of the X^3 -axis. A clarification of these points seems desirable.

Further remarks and investigations concerning the quantization of the gravity theories (3.3, 3.4) will be made in the course of this chapter after having explored the classical solutions into some detail.

3.2 General Solution to the Field Equations and All Extremals

As pointed out already in Sec. 2.3, given the present σ -model-like formulation of the gravity models the most straightforward way to determine the classical solutions is obtained by an appropriate choice of target space coordinates. In the present case let us choose coordinates

$$(C, X^\pm, X^3), \quad (3.11)$$

where C is the Casimir coordinate (3.5); the transition from the original X^i to these coordinates is well-defined for $X^\pm \neq 0$. In coordinates (3.11) the symplectic form is given by (2.44) on any of the level surfaces $C = \text{const}$,

except at the critical points described already in the previous section where it certainly vanishes. The usage of the coordinates (3.11) is more favorable than the one of Casimir-Darboux coordinates since it allows to cover larger parts of N and the charts with $X^+ \neq 0$ and $X^- \neq 0$ have a common overlap, constituting an atlas of N except for the line $X^+ = X^- = 0$.

In target space coordinates (C, X^+, X^3) the field equations take the simple form

$$dC = 0, \quad dA_C = 0, \quad A_+ = \frac{dX^3}{X^+}, \quad A_3 = -\frac{dX^+}{X^+} \quad (3.12)$$

as collected from (2.26, 2.31, 2.32) and (2.44). The local solution is obvious: $C = \text{const}$, $A_C = df$, $A_{+,3}$ as above, and f , X^+ , and X^3 are arbitrary functions, subject, however, to arbitrary gauge transformations (respecting the metric nondegeneracy). Via (1.7) we find the relation of the zweibein and the spin connection to the transformed connection:

$$\begin{aligned} e^- \equiv e_+ &= A_+ + 2 \exp(\alpha X^3) X^- A_C \\ e^+ \equiv e_- &= 2 \exp(\alpha X^3) X^+ A_C \\ \omega &= A_3 + 2 \exp(\alpha X^3) V A_C. \end{aligned} \quad (3.13)$$

From this we now read off the metric

$$g \equiv 2e^+ e^- = 4 \exp(\alpha X^3) dX^3 df + 4h(X^3, C) df df, \quad (3.14)$$

where h is the function defined in (3.7) and we suppressed introducing a symbol for the symmetrized tensor product.

For the torsion-free case $\alpha = 0$ we may now choose the gauge

$$X^3 = x^0, \quad X^+ = 1, \quad f = x^1/2 \quad (3.15)$$

respecting $X^+ \neq 0$ and $\det g \neq 0$. The metric (3.14) then takes the form

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & h(x^0) \end{pmatrix}. \quad (3.16)$$

The rest of the fields is determined trivially through (3.12, 3.13) and, using (3.7), $X^- = (1/2)h(x^0)$.

In the case $\alpha \neq 0$ it is the gauge

$$X^3 = \frac{1}{\alpha} \ln(\alpha x^0), \quad X^+ = 1, \quad f = x^1/2 \quad (3.17)$$

with $\alpha x^0 \in \mathbb{R}_+$ which again allows to write the metric in the form (3.16); here we have set somewhat sloppily

$$h(X^3(x^0)) =: h(x^0), \quad (3.18)$$

noting that the degree of the zeros of h is not changed by this substitution. The rest of the fields can again be read off directly from (3.12, 3.13) and (3.7).

The result above is obtained in an almost equally straightforward manner when one starts from the field equations in the original coordinates X^i and uses Polyakov's light-cone gauge in the form $e_0^- = 0, e_1^- = e_0^+ = 1$; the integration of the field equations turns out to be trivial in this gauge and the above found representatives result from a fixation of the residual local gauge freedom. Similarly appropriate is the use of the axial gauge $e_0^- = \omega_0 = 0, e_0^+ = 1$ [22].

The analogous shape of the solutions in target space coordinates (C, X^-, X^3) , valid on any patch with $X^- \neq 0$, is obtained most easily by applying the transformation

$$\begin{aligned} e^+ &\longleftrightarrow e^- \\ \omega &\longleftrightarrow -\omega \\ X^+ &\longleftrightarrow -X^- \\ X^3 &\longleftrightarrow X^3, \end{aligned} \quad (3.19)$$

to the above, since (3.19) reverses only the sign of the action integral (3.3) and therefore does not affect the equations of motion. Clearly the form of the metric remains that of (3.16) under the transformation (3.19). Although (3.19) is some 'cockscrewed' Lorentz transformation (resulting from the parametrization $\omega_{ab} = \varepsilon_{ab}$ and the fact that ε_{ab} is a 'pseudo-tensor'), it may be interpreted also as an active symmetry transformation mapping

different patches of the space time manifold onto each other; this fact will be useful in order to extend our local solutions to global ones.

Note also that in both charts $X^\pm \neq 0$ the whole solution is independent of x^1 . Thus there is a Killing field, $\frac{\partial}{\partial x^1}$, generating shifts in the x^1 -direction.

Simple zeros of X^a (both components) will occur when gluing together the above solutions. So, although possible in an analogous manner, it is not necessary to construct such local solutions here. For zeros of X^a of a higher degree the a -components of the field equations (2.17)

$$dX^a + \varepsilon^a_b (X^b \omega - V e^b) = 0 \quad (3.20)$$

together with the fact that the e^a are linear independent (metric nondegeneracy) show that the considered point is a singular point of P (cf. Eq. (3.6)). Since such points in the target space constitute an integral surface S by themselves and since the image of the worldsheet lies entirely in a symplectic leaf S (cf. Sec. (2.3)) solutions in the neighborhood of zeros of X^a of degree at least two have the form

$$X^a = 0, \quad X^3 = B_{crit} = \text{const.} \quad (3.21)$$

They are, furthermore, solutions of vanishing torsion and constant curvature,

$$De^a = 0, \quad d\omega = v'(B_{crit})\varepsilon \quad (3.22)$$

describing arbitrary deSitter space-time manifolds. The metric for such a solution can also be brought into the form (3.16) with $h(x^0) = c - v'(B_{crit})(x^0)^2$ where c is some meaningful constant of integration. This in turn determines the zweibein and spin connection up to Lorentz transformations.

To gain some feeling for gravitational solutions as well as to construct and analyze Penrose diagrams, it is standard to consider the movement of point particles within the space-time M determined by the (unperturbed) gravity solutions. For this purpose we couple $L_p[x(\tau)] = m \int \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) g_{\mu\nu}(x(\tau)) d\tau$ to our gravity action 3.3. Variation for $x(\tau)$ leads to the standard equation for extremals

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0 \quad (3.23)$$

where $\Gamma_{\mu\nu\rho} \equiv (g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu})/2$ are the Christoffel symbols. Disregarding the backreaction of the point particle on the metric ($m \rightsquigarrow 0$), we may combine this with the previously obtained solution for the metric, Eq. (3.16), to get ($h' \equiv dh(x^0)/dx^0$)

$$\ddot{x}^0 = -h' \dot{x}^1 (\dot{x}^0 + \frac{1}{2} h \dot{x}^1) \quad , \quad \ddot{x}^1 = \frac{1}{2} h' (\dot{x}^1)^2 . \quad (3.24)$$

Eq. (3.23) yields those curves which maximize the arclength s for nonnull extremals; the parameter τ is, up to linear transformations, the arclength itself [47]. (Obviously (3.23) is not form-invariant under reparametrizations, as is not L_p). In the torsion-free case these extremals coincide with autoparallels, satisfying, in the parametrization of (3.23), $\nabla_{\dot{x}} \dot{x} = 0$. In the case of nonvanishing torsion the extremals are autoparallel only with respect to the Christoffel connection $\Gamma_{\mu\nu\rho} = \omega_{\mu(\nu\rho)}$, where the brackets indicate symmetrization.

Since Γ is a metrical connection, any null-line $0 = g(\dot{x}, \dot{x})$ is a solution to (3.23, 3.24). In the charts of (3.16) these *null extremals* are:

$$x^1 = \text{const} , \quad (3.25)$$

$$\frac{dx^1}{dx^0} = -\frac{2}{h} \quad \forall x^0 \text{ with } h(x^0) \neq 0 \quad (3.26)$$

$$x^0 = \text{const} , \quad \text{if } h(x^0) = 0 . \quad (3.27)$$

Plugging these solutions back into (3.24), we can decide under which conditions they are complete (with respect to the affine parameter of (3.24)). For (3.25) and (3.26) the affine parameter is determined by $x^0 = a\tau + b$, $(a, b) = \text{const}$, thus these null extremals are complete, iff the coordinate x^0 extends to infinity into both directions of the charts in which the metric takes the form (3.16). For the models under study, this is the case for $\alpha = 0$, whereas in the case of nonvanishing torsion, $\alpha \neq 0$, the extremals are incomplete at $x^0 = 0$. Note, however, that in the latter case this line is a true singularity as curvature and torsion blow up there. The extremals (3.27), on the other hand, are complete for $h' = 0$ (multiple zero of the function h), but for $h' \neq 0$ we find $x^1 = -2(\ln|\tau + a|)/h' + b$, where $(a, b) = \text{const}$, so that they are incomplete at $\text{sgn}(h'(x^0)) x^1 \rightarrow +\infty$.

It will turn out that in our case the knowledge of the null extremals already suffices to find all Penrose diagrams. Nevertheless, also all other extremals can be found: Since $\frac{\partial}{\partial x^1}$ is a Killing field there is a constant of motion, $g(\frac{\partial}{\partial x^1}, \dot{x}) \equiv \dot{x}^0 + h\dot{x}^1 = \text{const}$; this holds in the case of nonvanishing torsion, too, where one can use the Christoffel connection for the proof (cf., e.g., [48]). Furthermore, we know that for nonnull extremals we may choose the length as affine parameter so that we obtain:

$$(dx^0 + hdx^1) = \text{const} \quad ds = \text{const} \sqrt{|2dx^0dx^1 + h(dx^1)^2|}.$$

The resulting quadratic equation has the solutions

$$\frac{dx^1}{dx^0} = \frac{-1 \pm \sqrt{\frac{c}{c-h}}}{h}, \quad c = \text{const} \quad (3.28)$$

$$x^0 = \text{const}, \quad \text{if } h'(x^0) = 0. \quad (3.29)$$

Certainly (3.28) is valid only when it is meaningful; the condition in (3.29) can be deduced directly from (3.24) and the fact the considered extremal is nonnull. It is straightforward to see that for the extremals (3.28)

$$ds = \frac{1}{\sqrt{|c-h|}} dx^0, \quad (3.30)$$

while the extremals (3.29) are obviously always complete.

3.3 Penrose Diagrams from Gluing

In this section we will provide the general rules of how to find the Penrose diagrams starting from any given metric of the form (3.16). This shall be done by means of a generic example, the function h of which is drawn in Fig. 2a. Having derived a simple building block principle, we will apply it to the Jackiw-Teitelboim model (deSitter gravity) (1.2), R^2 -gravity (1.3), and the Katanaev-Volovich model (1.4) at the end of this section.

In Fig. 2b we qualitatively depicted representatives of the null extremals (3.26, 3.27) corresponding to the metric of Fig. 2a. Any other solution (3.26) is obtained by shifting these curves along the x^1 -direction, a consequence of

the fact that $(x^0 = \text{const})$ -lines are Killing (isometry) directions within the chart x^μ . The other type of null lines, (3.25), are parallels to the x^0 -axis; from them we have drawn only those two which are the asymptotics to the representatives plotted of the former kind in the sector on the right.

It is well-known that any metric in two space-time dimensions is conformally flat, i.e. that by a change of coordinates it can be brought into the form $g = \exp(f(x))dx^+dx^-$; this can be achieved even on a global level, if the space-time manifold may be covered by one chart (and has no closed timelike curves). Within any of the sectors of Fig. 2b the diffeomorphism

$$x^+ = x^1 + f(x^0), \quad x^- = x^1 \quad (3.31)$$

with

$$f(x^0) \equiv 2 \int^{x^0} \frac{du}{h(u)} \quad (3.32)$$

provides such a transformation; it, however, breaks down at $h(x^0) = 0$. It may be difficult to write down explicitly the diffeomorphism that brings g into conformal form on all of the chart underlying Fig. 2b.

Fortunately, the explicit form of such a diffeomorphism need not be constructed. Similarly as in (3.31) it will be possible to choose x^1 as one of the light cone coordinates. The diffeomorphism will then have the effect of straightening the null extremals (3.26), leaving (3.27) as well as (3.25, i.e. $x^1 = x^- = \text{const}$, unmodified. Note that the (x^+, x^-) -chart cannot be all of \mathbb{R}^2 anymore; rather on the righthand side there will be some boundary because the null lines of type (3.26) do not intersect all null lines $x^- = \text{const}$. By means of a subsequent conformal diffeomorphism $x^- \rightarrow \tan x^-$ (and a similar one for x^+) the new coordinate chart covers only a finite region in \mathbb{R}^2 ; the result is drawn qualitatively in Fig. 2c. As indicated by the arrows, the boundary on the righthand side can be made straight by a conformal transformation in x^+ , by means of which one can also transform all rectangles into squares. The final building block for the Penrose diagram is obtained by turning the patch 45 degrees counter-clockwise (given our convention that a positive (negative) ds^2 corresponds to a timelike (spacelike) distance) and is depicted in Fig. 2d.

In the Figs. 2c,d we also included the lines of constant X^3 (= Killing directions), which in Fig. 2b have been the straight lines $x^0 = \text{const.}$ The function X^3 increases monotonously from the lefthand side,⁴ where it is $-\infty$, to the singularity on the righthand or upper side, where it gets $+\infty$. In our example Fig. 2a both of these lines $X^3 \rightarrow \pm\infty$ are complete, as the coordinate x^0 is unbounded in Fig. 2a (cf. the discussion following Eq. (3.25)); completeness will be indicated by boldfaced lines, incomplete singularities by thin solid lines (somewhat thicker than the lines of isometry), and horizons by dashed lines.

The square-shaped or triangular sectors are the regions with $(X)^2 \neq 0$ ($\Leftrightarrow h \neq 0$). As already noted above the triangular shape of the rightmost sector is due to the fact that any type (3.26) null extremal approaches a type (3.25) extremal assymptotically for $x^0 \rightarrow +\infty$ (which implies that the former kind of null lines does not intersect all of the latter kind). If we had instead $\int^{+\infty} \frac{dx^0}{h} = \pm\infty$ (e.g., $h = O(x^0)$), all null extremals would intersect each other exactly once and the triangular sector would be replaced by a quadratic one. If, on the other hand, e.g. $h \sim (x^0)^{n>1}$ for $x^0 \rightarrow -\infty$, then, vice versa, we would have to replace the quadratic sector on the left of Figs. 2c,d by a triangular one, again with a timelike boundary $X^3 = -\infty$. And so forth.

Finally, in Figs. 2b,c we have drawn some of the nonnull extremals (3.28) with $c=0$, i.e.,

$$\frac{dx^1}{dx^0} = -\frac{1}{h}, \quad (3.33)$$

as dotted lines. They are the unique extremals running through the corner-points of the $(X)^2 \neq 0$ sectors in all nonnull directions. Their length follows from (3.30):

$$s = \int_{\xi_1}^{\xi_2} \frac{dx^0}{\sqrt{|h|}}$$

where ξ_1, ξ_2 are the successive zeros of $h(x^0)$. Since in the chart (3.16) these extremals differ only by a shift in x^1 -direction, they all have the same length.

⁴For simplicity we assume $\alpha = 0 \Rightarrow x^0 = X^3$ here.

It is finite only at single zeros of $h(x^0)$:

$$s = \int \frac{dx^0}{\sqrt{|h|}} \sim \int^\xi \frac{dx^0}{(x^0 - \xi)^{\frac{n}{2}}} \rightarrow \begin{cases} < \infty & n = 1 \\ \infty & n \geq 2 \end{cases} . \quad (3.34)$$

Thus the cornerpoints of a sector between two single zeros of $h(x^0)$ are conjugate points.

Now any region with $(X)^2 \neq 0$ can be found in two charts of the kind (3.16), one with $X^+ \neq 0$ and one with $X^- \neq 0$, and it must be possible to establish a diffeomorphism (at least locally) between them. Such a diffeomorphism has to maintain X^3 and must map type (3.25) null extremals onto type (3.26) null extremals and vice versa. By this it is already uniquely determined as

$$\tilde{x}^0 = x^0 \quad , \quad \tilde{x}^1 = -f(x^0) - x^1 + \text{const} \quad (3.35)$$

where $f(x^0)$ is the function (3.32) introduced already above, and the integration constant has been written down explicitly, indicating the free choice of the origin of the x^1 -coordinate. It is easily seen that this diffeomorphism maps the whole sector where $(X)^2 \neq 0$. A subsequent Lorentz transformation is necessary to restore our gauge:

$$e^+ \rightarrow -\frac{2}{h}e^+ \quad e^- \rightarrow -\frac{h}{2}e^- \quad \omega \rightarrow \omega - d(\ln|h|). \quad (3.36)$$

The final result of this transformation (3.35, 3.36) is exactly that of (3.19). In the Penrose-diagrams the sectors are the squares (or the triangles), the second solution is obtained from the first by taking the mirror image (around an axis running diagonally through the sector, transversal to the Killing-direction!) and the gluing diffeomorphism (3.35, 3.36) amounts to patching the corresponding sectors together. Fig. 3 illustrates this process. By the above description gluing is unique, up to the constant, which, as long as only the universal covering is pursued, does not affect the solution.

We have yet to investigate the case (see Fig. 3) that after surrounding the point at the vertex of four diagrams the corresponding sectors of the first and the fourth diagram match. Shall they be identified?

The null extremals running into this vertex point are of the type (3.27). Now, if this zero of $h(x^0)$ is simple ($h'(x^0) \neq 0$) then the answer is yes: The transformation

$$\tilde{x}^0 = x^0 \quad \tilde{x}^1 = x^1 + \frac{f(x^0)}{2},$$

with $f(x^0)$ as before, which is on each $(X)^2 \neq 0$ sector a diffeomorphism, brings the metric into Schwarz-schild form

$$ds^2 = -\frac{1}{h(\tilde{x}^0)}(d\tilde{x}^0)^2 + h(\tilde{x}^0)(d\tilde{x}^1)^2,$$

and the Kruskal extension (cf. e.g. [49]), plus simultaneous Lorentz transformation, reveals the vertex point as regular interior point (a saddle point of X^3 , with $X^a = 0$). The four adjacent sectors then constitute one single sheet.

However, for zeros in $h(x^0)$ of a higher degree this procedure fails. And, in fact, not only the null extremals 3.27 but also the general extremals 3.33 running towards this point are then complete (cf. Eq. 3.34). Thus the vertex point has to be taken out of consideration. Also, to obtain the universal covering we must not identify the diagrams but continue the gluing indefinitely.

The principle of how to construct *the* Penrose diagram corresponding to any function h in (3.16) should be clear now. The number (and kind) of zeros of this function determines the number of squares in a fundamental building block,⁵ the end of which is either a square or a triangle, depending on the asymptotic behavior of h . The complete Penrose diagram is then obtained by straightforward (pictorial) gluing.

Let us come to the announced examples, starting with deSitter gravity (1.2). Since all values $\Lambda \neq 0$ yield (basically) equivalent Penrose diagrams, we will set $\Lambda := 1$ in the following. We then get a one-parameter family of functions h , $h(x^0) = C - (x^0)^2$, parametrized by the Casimir constant $C = (X)^2 + (X^3)^2$ (cf. Fig. 4).

For any $C > 0$ this curve $h(x^0)$ has two simple zeros, leading to one square within the fundamental building block. Asymptotically we have $h \sim -(x^0)^2$, so that adjacent to the square there will be a triangle at each side, the

⁵The case that h has no zeros at all will be discussed in the applications below.

boundaries of which, $X^3 = \infty$ resp. $X^3 = -\infty$, are spacelike and complete. Gluing leads to the ribbon-like diagram shown in Fig. 5a. (For the other sign of the cosmological constant, $\Lambda = -1$, we get the same diagram for $C < 0$, rotated, however, by 90 degrees, as the infinity is timelike then).

For $C = 0$ we get no square, but only two triangles. The corresponding Penrose diagram is plotted in Fig. 5b.

For $C < 0$ the function h has no zeros (Fig. 4). We therefore may apply directly the diffeomorphism (3.31) with the function (3.32), which in the present case can be written in terms of elementary functions:

$$f(x^0) = -\frac{2}{\sqrt{-C}} \arctan\left(\frac{x^0}{\sqrt{-C}}\right). \quad (3.37)$$

The resulting region (x^+, x^-) is again a ribbon (Fig. 5c); but this time without any internal structure, as the Killing lines $X^3 = x^0 = \text{const}$ become the parallels $x^+ - x^- = \text{const}$ in the present case.

It is straightforward to see that Fig. 5c depicts the Penrose diagram for any (negative) function h without zeros which diverges at the infinity. Clearly there is no conformal diffeomorphism which maps this ribbon into a finite region. Let us remark also that the Penrose diagrams for constant curvature in two dimensions are quite different from the ones in four dimensions (cf., e.g., [49, 48]).

Let us now turn to the second example: R^2 -gravity. We have already studied the behavior of the corresponding function h when discussing the quantum theory in Sec. (3.1, cf. Fig. 1. It is completely straightforward to construct the Penrose diagrams according to the above rules. The result is depicted in Fig. 6. For nonnegative Λ there are in addition to those diagrams also infinite bands for the constant curvature solutions $d\omega = \pm 2\sqrt{\Lambda}\varepsilon$ (cf. Eq. (3.22); in the diagram Fig. 1 they are located at the curve **R2,3,4**.

The third example is the Katanaev-Volovich model (1.4). Its potential is

$$V^{KV} = (X^3)^2 - \Lambda + \frac{\alpha}{2}(X)^2 \quad (3.38)$$

leading to the Casimir function

$$C^{KV} = \frac{2}{\alpha} \exp(\alpha X^3) \left(V^{KV} - \frac{2X^3}{\alpha} + \frac{2}{\alpha^2} \right) - \frac{4}{\alpha^3} + \frac{2\Lambda}{\alpha}. \quad (3.39)$$

From this and (3.7,3.17,3.18) one can determine the function $h(x^0)$. The coordinate transformation $\alpha x^0 \rightarrow x^0$, $x^1/\alpha \rightarrow x^1$, constituting a residual gauge freedom of (3.16), combined with the rescaling

$$\tilde{\Lambda} := \alpha^2 \Lambda, \quad \tilde{C} := \alpha^3 C^{KV} - 2\tilde{\Lambda} + 4 \quad (3.40)$$

brings it into the simpler form ($x^0 \in \mathbb{R}_+$)

$$h(x^0) = \frac{1}{\alpha} \left\{ \tilde{C} x^0 - 2(x^0)^2 [(\ln x^0 - 1)^2 + 1 - \tilde{\Lambda}] \right\}. \quad (3.41)$$

It shows that, up to its sign, α does not influence the causal structure of the KV-model.

The function h always has a zero at $x^0 = 0 \Leftrightarrow \alpha X^3 = -\infty$, which is simple for $\tilde{C} \neq 0$. It corresponds to an incomplete null-infinity. On the other boundary of the coordinate patch, $\alpha X^3 = +\infty$, we find $\alpha h \sim (x^0)^2 \ln^2 x^0$, which shows that the Penrose diagrams have a complete triangular sector at this end.

To study the number and kind of zeros of the function (3.41) for positive values of x^0 , one is well advised to change variables according to $y = \ln x^0$, being left with the *equivalent* analysis of the number and type of zeros of the function $f(y) = \tilde{C} \exp(-y) - 2[(y - 1)^2 - \tilde{\Lambda} + 1]$ within $y \in \mathbb{R}$. (Actually $y = \alpha X^3$ and $f(\alpha X^3) \equiv \alpha^3 (X^3)^2 (X^3)$). In any case the analysis yields 11 qualitatively different cases depending on the parameters \tilde{C} and $\tilde{\Lambda}$:

- G1,2** : no zeros of h and thus also of $(X)^2$
- G3** : one single zero at $X^3 = B$
- G4** : one triple zero at $X^3 = 0$
- G5,6** : one double zero at $X^3 = \text{sgn}(\alpha) \sqrt{\tilde{\Lambda}}$
- G7** : one double zero at $X^3 = -\text{sgn}(\alpha) \sqrt{\tilde{\Lambda}}$ and one single zero at $X^3 = B_1$
- G8,9** : two single zeros at $X^3 = B_2$ and $X^3 = B_1$
- G10** : one single zero at $X^3 = B_3$ and one double zero at $X^3 = \text{sgn}(\alpha) \sqrt{\tilde{\Lambda}}$
- G11** : three single zeros at $X^3 = B_3, B_2$, and B_1 ,

where $B_{3(1)} < -\sqrt{\tilde{\Lambda}} < B_2 < +\sqrt{\tilde{\Lambda}} < B_{1(3)}$ for $\alpha > (<) 0$ and $-\infty < B < +\infty$. An overview is provided by Fig. 7.

Via (3.6,3.4,3.38) the critical values of X^3 are easily determined to be

$\pm\sqrt{\Lambda}$; the corresponding value of the (rescaled) Casimir function \tilde{C} is

$$\tilde{C}_{crit} \equiv \tilde{C}_{deSitter} = -4 \exp\left(\pm\sqrt{\tilde{\Lambda}}\right) \left(\pm\sqrt{\tilde{\Lambda}} - 1\right), \quad (3.42)$$

which marks the curve **G5,6,10,4,7** of Fig. 7 and simultaneously the deSitter solutions $De^a = 0$, $d\omega = \pm\sqrt{\Lambda}\varepsilon$ (cf. Eqs. (3.21,3.22)). The cases **G1,2**, as well as **G5,6** and **G8,9**, differ by the kind of zero of h at $x^0 = 0$; this has its impact on the completeness of the corner point. It is now straightforward to draw the Penrose diagrams of the KV-model. The result is depicted in Fig. 8 for $\alpha > 0$; the diagrams for $\alpha < 0$ are obtained by rotating these by 90 degrees.

The numbering **G1-11** has been chosen as in [24], where the Penrose diagrams Fig. 8 have been constructed first. It should be noted, however, that our procedure to obtain these diagrams is incomparably faster than the one of [24]. The main reason is that the local solutions used in [24] (resulting also from ours through the diffeomorphism (3.31)) are valid only in coordinate patches which are part of ours (the sectors $(X)^2 \neq 0$); they had to be glued along their border, which entailed lengthy considerations of the asymptotic behavior. In our gauge, instead, the charts overlap and simply have to be matched together. As a consequence we also could prove that all the solutions of (1.4), (and in fact also of (3.3,3.4) with an, e.g., analytic potential v) are analytic. Also, from (3.16) the existence of a Killing field is immediate.

Concluding we remark that for many of the Penrose diagrams, such as, e.g., for **G3**, **G9**, it is possible to find also global coordinates displaying explicitly the analyticity of g . Examples for this might be given elsewhere.

3.4 All Global Solutions and a Comparison with the Quantum Theory: The Example of the Katanaev-Volovich-Model

In the following we shall give an account of global solutions obtained by factoring the universal coverings by a discrete transformation group. In this way one obtains *all* global solutions. Among these are the solutions with cylindrical topology. Keeping track of all diffeomorphism (and Lorentz) invariant quantities characterizing such solutions, we get some insight into the reduced phase space (RPS) of the theory as defined on the cylinder. Note that the Penrose diagrams are labelled only by one constant, namely the value of the Casimir function C (beside, of course, the coupling constants fixed in the Lagrangian). Since the result of a symplectic reduction is again a symplectic space, with some care we will be able to find a second continuous parameter resulting from the compactification.

In the sections 2.4 and 3.1 we have studied already the quantum theory of the space of gauge-inequivalent solutions with cylindrical topology. Certainly, a comparison of this quantum theory with the classical space-time manifolds that are subject to this quantization is worth an investigation. The comparison will be seen to give rise to arguments in favor of the quantization scheme employed, but there will arise also arguments questioning it. On the one hand we will find that the topology of the RPS fits quite perfectly to the arguments of the wave functions (2.43); in particular, the quantum numbers n_C are in one-to-one correspondence with the minimal number of building blocks intersecting a noncontractible loop on the cylinder. On the other hand, a considerable portion of the solutions in the RPS are incomplete, they have closed timelike curves, and for some of them it would be more natural to be reckoned among other space-time topologies than the cylindrical one.

Most of these questions will be analyzed at the example of the Katanaev-Volovich model (1.4), but the discussion transfers in an obvious way to the general model (3.3,3.4).

Let us start with classifying all possible 'compactifications' of the Penrose diagrams. They are obtained by factoring out a discrete transformation group from the universal covering solutions. Any such transformation must of course preserve the functions X^3 and X^a . Hence the sectors with $(X)^2 \neq 0$ must be mapped as a whole onto corresponding ones (i.e., with the same range of X^3).

Within such a sector we have already discovered a Killing field ($\frac{\partial}{\partial x^1}$ in the coordinates (3.16)). The transformation generated by it is in local charts (3.16) a shift of a certain amount in the x^1 -direction. The gluing diffeomorphism (3.35, 3.36) shows that such a transformation extends uniquely onto the whole universal covering, and that it is in all charts represented as an x^1 -shift of the same amount (but on part of them in the opposite direction!). We will call these transformations simply 'Killing-shifts'. In the Penrose diagram such a Killing-shift shows as a distortion along the lines of constant X^3 .

A further transformation of a sector onto itself is exchanging the two types of null-extremals. This is exactly the gluing diffeomorphism of (3.35, 3.36). It can also be described as a 'reflection' at one of the extremals (3.33).

Any admissible transformation can thus be separated into a combinatorial part — a certain permutation of the $(X)^2 \neq 0$ sectors and their possible reflections — and one real parameter describing the Killing-shift. It is also true that any transformation of the universal covering is already fully determined by the image of only one sector. The remaining investigations shall be performed ad hoc at the example of the Katanaev-Volovich model now.⁶

G1,2: The only transformations are reflections and Killing-shifts. Since a reflection has fixed points (an extremal of type (3.33) as symmetry-axis) it has to be ruled out. The only discrete subgroups of Killing-shifts are the infinite cyclic groups generated by one shift. The factor space is then clearly a cylinder. In the coordinates (3.16) it can be obtained by cutting out a strip parallel to the x^0 -axis and gluing it together along the frontiers. The width

⁶The numbering below refers to the numbering of the Penrose diagrams in the Figs. 7,8.

of this strip (i.e., of the generating shift) is proportional to the length of a constant curvature path running once around the cylinder. (Remember that in this model $X^3 \propto$ Ricci scalar, on-shell). Hence for any value $C \geq 0$ we get a set of distinct solutions parametrized by their size (any positive real number).

G3: As before reflections and also the inversion at the central saddle point must be dropped. We could again try a cylinder-solution, but there occur some problems, if one adopts a standard gravity point of view: Not only is there a closed null-extremal ($x^0 = \text{const}$, at $(X)^2(x^0) = 0$), but other extremals approach it asymptotically, winding around the cylinder infinitely often while having only finite length. This situation resembles strongly the Taub-NUT space or rather its two-dimensional analog as described by Misner [50] (cf. also [49]). As explained there an extension is possible, if one abandons the Hausdorff property. The net result can be described as two concentric cylinders attached to each other at the (closed) line $(X)^2 = 0$; those extremals which previously had been incomplete are now continued at the 'other sheet' previously not included. In fact, the resulting extended solutions occur naturally as factor space of the universal covering.

On the other hand, from the purely field theoretic point of view, there is no notion of geodesic (or extremal) completeness; the solutions on the cylinder are perfectly analytic everywhere on the cylinder (cf. Eqs. (3.16,3.15,etc.) taking the x^1 -coordinate as periodic now), and there also is the metric induced circumference as the variable 'conjugate' to $C < 0$. (Changing the length of periodicity of x^1 , one can change the length of an $(X^3 = \text{const})$ -line; since such a circumference is, for any fixed value of X^3 , a gauge-independent quantity, it may represent the second variable besides C).

Taking together the cylindrical solutions for **G1-G3**, from the field theoretic point of view we find a perfect coincidence between the quantum theory of the KV-model with $\Lambda < 0$ (cf. Fig. 7) and the corresponding RPS, which is just a plane (identifying the four possible cylinders for $C < 0$ by means of the discrete symmetry transformation (3.19)). Nevertheless, according to the above considerations, it seems more natural to regard the cylindrical so-

lutions for $C < 0$ as a *part* of the extended nonHausdorff object described above. A possible, somewhat speculative interpretation of the quantum theory for $\Lambda < 0$ could then be that wave functions with support on $C \geq 0$ correspond to cylindrical space-times, their counterparts with $C < 0$ correspond to these 'four-sheet' space-times, and quantum processes mixing these vector-subspaces describe topology changes. The other way out might be to just regard the Hamiltonian methods used so far to be not sophisticated enough to cope with such problems of quantum gravity as geodesic completeness and different topologies.

In the following we will continue studying all factorizations possible in the remaining solutions **G4 - G11**. Since Taub-NUT like solutions exist for *all* of these cases, we will not mention them any further.

G8,9: Since this solution is an infinite ribbon of equal building blocks, it is possible to factor out a shift of a number of blocks to obtain a cylinder. Furthermore, while pure reflections (and inversion at a saddle point) have to be ruled out, a 'vertical' reflection plus 'horizontal' shift will work and it yields a Möbius-strip.

Let us now come to find the second phase space parameter (for the cylindrical solutions). It was already pointed out that the generating shift-transformation has an additional real parameter, the Killing-shift component. We have yet to show that different Killing-shift values yield inequivalent solutions: In the previous section it was proved that the saddle points are conjugate points and the extremals running between them are those of (3.33). They run through the saddle points in all directions between the two null-directions. A Killing-shift shifts them sideways, altering the angle of their tangent. One can now start from a saddle point in a certain direction along a spacelike extremal. This extremal will eventually return to the original point, but due to a Killing-shift its tangent at the return may be tilted against that at the start. Since this tilt can be expressed in terms of a shift of the x^1 -coordinate in the chart (3.16), it is the same for *all* such extremals. Especially, there is one solution without tilt. Thus, besides the parameter C , the cylindrical solutions are parametrized by a positive integer (number

of copies or ‘building blocks’) and a real constant parametrizing the tilt.⁷

It may seem that for the Möbius-strip one might also have this continuous parameter. However, in contrast to the former examples sectors are occasionally identified with their mirror images. This has the consequence that on the factor space a Killing-shift cannot be defined consistently. Furthermore there is exactly one extremal which has the same tangent even at the first return. A Killing-shift component of the transformation only results in a different choice of this special extremal and thus (by a coordinate change) leads to equivalent solutions. Hence, besides C , the Möbius-strip solution is only parametrized by a positive integer (number of copies).

G5,6: Again reflections cannot be used. As for **G8,9** we get cylinders parametrized by a positive integer (the number of copies) and a real number (Killing-shift), but this time no Möbius-strips. In contrast to G8,9 there is no such nice description of the Killing-shift parameter, because we have in general no closed extremals. One can, however, take a series of null extremals, zigzagging around the cylinder between two values of X^3 (or in this case even an oscillating spacelike extremal) and interpret the failure to be closed (i.e., the distance between starting- and endpoint on this $X^3 = \text{const}$ line) as measure for the Killing-shift.

G4: As before this solution is an infinite ribbon of equal building blocks (although this time not straight but winding around the central point). Thus one obtains again cylinders of a certain number of copies. When passing once around such a cylinder, however, the light cone tilts upside down n times, where n is the number of copies involved. Thus we have got an n -*kink*-solution. Hence the nonTaub-NUT solutions are cylinders parametrized by a positive integer (the number of copies = number of kinks) and a real number (Killing-shift).

G7,10,11: These cases are slightly more complicated. Evidently G7 and G10 give rise among others to a cylinder with hole(s) and G11 to a torus

⁷If we had chosen $\alpha < 0$ the whole Penrose diagram would have to be rotated by 90 degrees. The above extremals would then be timelike and the tilt at the return could be interpreted nicely as acceleration along one journey around the cylinder.

with hole(s). Furthermore, each hole yields an additional real parameter (characterizing the re-identification after surrounding this hole) such that the n -hole cylinder or the n -hole torus has $n + 1$ real parameters.

But even a series of proper (yet slightly pathological) cylinders can be obtained: The universal covering covers the cylinder resp. the torus with hole. The group of cover-transformations is isomorphic to the fundamental group of the covered space, in this case the free group with two generators. It contains only admissible (isometric etc.) transformations but not all of them (e.g. reflections are missing). Any subgroup yields a factor space, especially a cyclic subgroup yields a cylinder.

To speak in pictures: The generator of this cyclic subgroup defines a path in the universal covering. Now the end-sectors of the path (i.e. of the corresponding ribbon) are identified and at all other junctions the solution is extended infinitely without further identifications. Thus a topological cylinder (although with a terribly frazzled frontier) is obtained.

For reasons of completeness one should treat also the deSitter solutions (3.21,3.22) of the KV-model, corresponding to the Casimir values (3.42). Partially this gap will be closed in the next section, where we will concentrate on all cylinder-solutions of the Jackiw-Teitelboim model. Since, however, the momenta are constant all over the space-time manifold in this case, there might be additional possibilities to compactify the deSitter solutions of the KV-model as compared to the ones found in the regular sector of the JT-model. We shall not investigate this here further.

Note also that from the quantum theory point of view the deSitter solutions, as well as the cylinder solutions **G4,5,6,7**, and **G10**, should be more or less negligible as they correspond to only one value of C (cf. also Fig. 7). The space of orbits, furthermore, is nonHausdorff exactly at these solutions.

Again the discrete indices present in (2.43) fit to the cylinder-solutions found above. However, at least in the case of **G11** I feel some unease with these 'cylinders'. And now one also cannot reinterpret the wave functions of this sector in the RPS to actually describe the natural factor space of a torus with hole, since the latter is parametrized by *three* continuous quantities.

3.5 Symmetries, Metric-Nondegeneracy, and Kinks: The Example of the JT-Model

One of the features of the Ashtekar formulation of 4D gravity usually considered as an advantage is that the formulation is well-defined also for configurations corresponding to degenerate metrics. In particular, the field equations and the symmetry transformations are equivalent to the usual Einstein formulation only for $\det g \neq 0$. It is the purpose of this section to investigate a similar relationship between two formulations of the symmetry content of the class of gravity models considered in this work which are also equivalent for nondegenerate configurations only.

According to (2.19) and (1.7) the gravitational symmetries, i.e. diffeomorphisms and local Lorentz transformations, can be identified with the symmetry transformations (2.15, 2.16) on-shell, if and only if $e := \det e_\mu^a \neq 0$ ($\Leftrightarrow \det g \neq 0$). As is obvious from the paragraph of Eq. (2.23) and the one following it, the symmetries (2.15, 2.16) coincide with the Hamiltonian symmetries; factoring out the former from the space of solutions to the field equations (for cylindrical space-time topology) is equivalent to a symplectic reduction⁸ yielding the reduced phase space (RPS) of the theory. The latter was the space subject to quantization in sections 2.4 and 3.1.

In this section let us compare the following two moduli spaces:

1) The space of (C^∞ -)solutions to the field equations on the cylinder modulo the symmetry transformations (2.15, 2.16). As noted above this space is equivalent to the standard RPS.

2) The same space of solutions, but excluding from it all configurations which correspond to a somewhere degenerate zweibein; only those solutions are identified which are related by gravitational symmetries.

There are basically two reasons that could give rise to a difference between these two spaces. Firstly, there could exist gravitationally unacceptable solu-

⁸A symplectic reduction is performed by implementing a system of first class constraints $G^i = 0$ in a phase space, factoring out the Hamiltonian flow of the G^i on this subspace subsequently.

tions to the field equations which are not gauge related to any gravitationally acceptable one. In this case the RPS of **1** contains points not included in **2**. Secondly, symmetry orbits of **1** could be cut into pieces by regions (in the space of solutions) which have somewhere (in M) degenerate zweibeins. In this case there are several points in **2** corresponding just to one point in **1**.

The first of these two mechanisms occurs for the Euclidean formulation of the gravity theories. Only bundles which have a Chern class that coincides with the one of the canonical (tangential) bundle on M will yield nowhere degenerate metrics. E.g. on a sphere the trivial bundle will in no way yield nondegenerate metrics; one necessarily will have to introduce at least two charts with a nontrivial gauge transformation (2.15, 2.16) on their overlap.

The second mechanism occurs in the case of Minkowskian gravity on $M = S^1 \times \mathbb{R}$ and shall be illustrated by means of a simple example: Take on the one hand the real line \mathbb{R} (\sim space of all solutions) and as the symmetry transformations translations with generator $T_1 = \partial/\partial q$ (\sim symmetries (2.15, 2.16)). Take on the other hand $\mathbb{R} - \{0\}$ ($q = 0 \sim e = 0$) modulo the transformations generated by $T_2 = qT_1$ (\sim gravitational symmetries). The degenerate point $q = 0$ is gauge related to $q \neq 0$ with respect to T_1 ; thus we do not have a problem of the first kind. However, the symmetry orbit of T_1 , which reduces \mathbb{R} to a point, is cut into two pieces by the fixed point $q = 0$ of the T_2 -transformations. On *all* of \mathbb{R} there are three gauge orbits of T_2 , \mathbb{R}^+ , \mathbb{R}^- and $\{0\}$, corresponding to three point in the factor space. Even if the point $q = 0$, where the correspondence between the infinitesimal form of the two symmetry transformations breaks down, is now removed, we end up with different results.

We will show in the present section that indeed eliminating the solutions with $\det g = 0$, the gauge orbits of (2.15, 2.16) split into components not smoothly connected to each other. Solutions from different components of the same gauge orbit are not related by gravitational symmetry transformations (since obviously $\det g = 0$ is a fixed point under the latter). They correspond to space-time manifolds with different kink-number. Although it is possible to parametrize the gauge orbits of the constraints globally for all models

(3.3, 3.4), we will restrict the analysis below to the case $V = X^3$, i.e. to the Jackiw-Teitelboim model, in which case this parametrization is particularly simple. As already noted in the introduction for this V the action (3.3) can be rewritten *identically* as (2.25) with Lie algebra $so(2, 1) \sim sl_2$ [6]; thus the gauge orbits are the standard ones of a nonabelian gauge theory.

The Penrose diagrams of the Jackiw-Teitelboim model, Fig. 5, do not allow for any complete kink solution. This coincides with the fact that all the kink solutions we will obtain as representatives of the different parts of the gauge orbits cannot be geodesically completed. Nevertheless, from the field theoretic point of view they constitute perfect C^∞ solutions on M (with an everywhere nondegenerate metric), which cannot be transformed into each other by means of the gravitational symmetries but are gauge related by sl_2 transformations. The fact that these incomplete kink solutions are eliminated automatically when using the Hamiltonian constraints G^i as symmetry generators could be regarded as an advantage of the formulation over some other formulation which strictly implements only gravitational symmetries. On the other hand this coincidence could be regarded also as purely accidental, two mistakes cancelling each other, an argumentation that can gain some support from the observations made in the preceding section.

The investigation below provides also some insight into the topology of the RPS. In particular it will be seen to be not Hausdorff. We suppose that this inevitably leads to some ambiguity in the quantization (in addition to the choice of polarization).

In the following we will determine the Hamiltonian RPS by means of the equivalent group theoretic formulation, which allows us to use comparatively simple fiber bundle methods. Since the constraints can generate only gauge transformations connected to the identity and since large gauge transformations are in one-to-one correspondence to the first fundamental group $\Pi_1(G)$ of the gauge group G , the gauge group G we have to use is the universal covering group of $SL(2, \mathbb{R})$, denoted by $\widetilde{SL}(2, \mathbb{R})$. The latter, however, has no faithful (finite dimensional) matrix representation. This technical obstacle is overcome by first determining the factor space using the gauge group

$PSL(2, \mathbb{R}) \sim SL(2, \mathbb{R})/\{1, -1\}$, withdrawing then the additional identification by applying large gauge transformations to the obtained representatives. In this way we will obtain representatives of the $\widetilde{SL}(2, \mathbb{R})$ gauge theory, which from the group theoretical point of view describe equivalence classes of paths in $PSL(2, \mathbb{R})$ parametrized by x^1 . All of these will correspond to solutions with $e \equiv 0$. Parametrizing the $\widetilde{SL}(2, \mathbb{R})$ -orbits through these representatives, we then will find an infinity of gravitationally inequivalent ways to ensure $e \neq 0$ everywhere on the cylinder M .

For reasons of explicitness let us choose a basis T^i , $i \in \{+, -, 3\}$, of the sl_2 -algebra which satisfies $[T^-, T^+] = T^3$, $[T^\pm, T^3] = \pm T^\pm$. A real matrix representation of this is provided by $T^\pm = (\sigma^1 \pm i\sigma^2)/2\sqrt{2}$, $T^3 = -\sigma^3/2$, where the σ 's are the standard Pauli matrices. In this way we can represent all fields by matrices through $A = A_i T^i$ and $X = X_i T^i = X^i T_i$, where the indices shall be raised and lowered by means of half of the Killing metrik κ : $\frac{1}{2}\kappa_{+-} = \eta_{+-} = 1$, $\frac{1}{2}\kappa_{33} = 1$. With (1.7) one thus has

$$A = \begin{pmatrix} -\omega/2 & e^-/\sqrt{2} \\ e^+/\sqrt{2} & \omega/2 \end{pmatrix}, \quad X = \begin{pmatrix} -X^3/2 & X_+/\sqrt{2} \\ X_-/\sqrt{2} & X^3/2 \end{pmatrix}, \quad (3.43)$$

and the action (1.6) coincides with (3.3) for $V + X^3$. The factors two and $\sqrt{2}$ above have been introduced so as to avoid any conflict with the conventions chosen in the gravity formulation. Note also the following identity for the Casimir invariant: $C = X^i X_i = 2\text{tr}(XX) = -4\det X$.

The group \mathcal{G} of the symmetry transformations we consider in the first stage is the group of smooth mappings from the cylinder into $PSL(2, \mathbf{R})$:⁹

$$\mathcal{G}_{PSL(2, \mathbb{R})} = \{g : S^1 \times \mathbf{R} \rightarrow PSL(2, \mathbf{R})\} \quad (3.44)$$

The equations of motion, which in the matrix notation introduced above take the form

$$F = 0, \quad dX + [A, X] = 0, \quad (3.45)$$

yield the connection to be flat and the field X to be covariantly constant. Up to gauge transformations a flat connection A on a cylinder is determined by

⁹There are no nontrivial principal G -bundles on a cylindrical base manifold, iff the chosen structure (gauge) group G is connected.

its monodromy $M_A = \mathcal{P} \exp \oint A \in PSL(2, \mathbf{R})$ generating parallel transport around the cylinder (\mathcal{P} denotes path ordering and the integration runs over a closed curve \mathcal{C} winding around the cylinder once). As the exponential map is surjective on $PSL(2, \mathbb{R})$, any monodromy matrix can be generated by a connection of the form $A = A_1 dx^1$ where A_1 is constant:

$$A = \begin{pmatrix} z & y+t \\ y-t & -z \end{pmatrix} dx^1, \quad t, y, z \in \mathbf{R}. \quad (3.46)$$

Constant gauge transformations act on A via the adjoint action leaving the determinant $t^2 - y^2 - z^2$ invariant and may be interpreted as Lorentz transformations in the three dimensional Minkowski space (t, y, z) . Hyperbolic, elliptic and parabolic elements, respectively, in the Lie algebra correspond to spacelike, timelike, and lightlike vectors, respectively, in this Minkowski space. By Lorentz transformations in the (t, y, z) plane they can be brought into the form:

$$\begin{aligned} A^{hyp} &= \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} dx^1, & A^{ell} &= \begin{pmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{pmatrix} dx^1, \\ A^{par} &= \begin{pmatrix} 0 & 0 \\ \pm 1 & 0 \end{pmatrix} dx^1 \end{aligned} \quad (3.47)$$

with $\alpha, \vartheta \in \mathbf{R}$ and the identification $\alpha \sim -\alpha$. Exponentiation yields the monodromy matrices

$$\begin{aligned} M_{A^{hyp}} &= \begin{pmatrix} \cosh 2\pi\alpha & \sinh 2\pi\alpha \\ \sinh 2\pi\alpha & \cosh 2\pi\alpha \end{pmatrix}, & M_{A^{ell}} &= \begin{pmatrix} \cos 2\pi\vartheta & \sin 2\pi\vartheta \\ -\sin 2\pi\vartheta & \cos 2\pi\vartheta \end{pmatrix}, \\ M_{A^{par}} &= \begin{pmatrix} 1 & 0 \\ \pm 2\pi & 1 \end{pmatrix}, \end{aligned} \quad (3.48)$$

inducing the further identification $\vartheta \sim \vartheta + 1/2$ in the elliptic sector (remember $\oint dx^1 = 2\pi$ and $1 \sim -1$). The integration of the second Eq. (3.45) gives $X(x^0, x^1) = X(x^0, x^1 + 2\pi) = M_A X(x^0, x^1) M_A^{-1}$. Thus choosing a connection from (3.47), $X(x)$ has to commute with the corresponding monodromy matrix and consequently with the connection itself. Using (3.45) again, one

finds $X(x)$ to be constant. We obtain:

$$\begin{aligned} X^{hyp} &= \begin{pmatrix} 0 & c_1 \\ c_1 & 0 \end{pmatrix}, & X^{ell} &= \begin{pmatrix} 0 & c_2 \\ -c_2 & 0 \end{pmatrix}, \\ X^{par} &= \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \end{aligned} \quad (3.49)$$

where the c_i are arbitrary real parameters. Note, however, that due to $(\alpha, c_1) \sim (-\alpha, -c_1)$ the hyperbolic sector of the $PSL(2, \mathbb{R})$ -RPS is a cone.

In the case $A = 0$ (corresponding to $\alpha = 0$ or $\vartheta = 0$, respectively, in (3.47)) $X(x)$ is constant, too, but it is not restricted by its commutator with the monodromy matrix. It is, however, subject to constant gauge transformations, as they leave $A = 0$ invariant. Considerations similar to those above show that also in this case gauge representatives of the solutions are given by (3.49) with $c_3 = \pm 1$ and the identification $c_1 \sim -c_1$.

(3.47, 3.49) with $(\alpha, c_1) \sim (-\alpha, -c_1)$ together with the $A = 0$ -sector give a complete parametrization of the reduced phase space of the $PSL(2, \mathbf{R})$ -gauge theory. As the configuration variable M_A is compact in the elliptic sector of the RPS, the corresponding conjugate variable C will have a discrete spectrum for $C < 0$ in the quantum domain; furthermore, there will exist some Θ -angle within this spectrum as the RPS is not simply connected. (Θ will label the irreducible representations of the fundamental group of the RPS which is \mathbf{Z} ; thus $\Theta \sim \Theta + 2\pi$). Indeed, these expectations have been confirmed in [10].

As already indicated above, the group of gauge transformations $\mathcal{G}_{PSL(2, \mathbf{R})}$ is not connected; rather it consists of an infinite number of components not smoothly connected to each other: $\Pi_0(\mathcal{G}) = \Pi_1(PSL(2, \mathbf{R})) = \mathbf{Z}$. A complete set of representatives for the components of $\mathcal{G}_{PSL(2, \mathbf{R})}$ is given by

$$g_{(n)} = \begin{pmatrix} \cos(nx^1/2) & \sin(nx^1/2) \\ -\sin(nx^1/2) & \cos(nx^1/2) \end{pmatrix}, \quad n \in \mathbf{Z}. \quad (3.50)$$

Parametrizing the phase space as in (3.47) - (3.49) we also implemented these gauge transformations. The action of the group elements $g_{(n)}$ on the

connections (3.47) gives in the hyperbolic sector

$$\begin{aligned} A_{(n)}^{hyp} &= \begin{pmatrix} \alpha \sin(nx^1) & \alpha \cos(nx^1) + n/2 \\ \alpha \cos(nx^1) - n/2 & -\alpha \sin(nx^1) \end{pmatrix} dx^1 \\ X_{(n)}^{hyp} &= c_1 \begin{pmatrix} \sin(nx^1) & \cos(nx^1) \\ \cos(nx^1) & -\sin(nx^1) \end{pmatrix}. \end{aligned} \quad (3.51)$$

An analogous result is obtained in the parabolic sector. In the elliptic sector the $g_{(n)}$ generate a transformation $\vartheta \rightarrow \vartheta + n/2$. They are responsible for the previous identification $\vartheta \sim \vartheta + 1/2$, which now is removed again.

In this way we have found a complete parametrization of the RPS of the Jackiw-Teitelboim model. It agrees perfectly with the quantum mechanical system obtained for it: Obviously $X_{(n)}^{hyp}$ are representatives of the first homotopy of the coadjoint orbit $C = 4(c_1)^2 > 0$ and the integer n can be chosen to coincide with the discrete index found in (2.43) obviously coincides with the integer n present in the parametrization of the hyperbolic sector ($C > 0$) of the RPS. Note also that we had to use (at least) two charts to depict the Jackiw-Teitelboim wave functions in the form (2.43); they correspond to the two signs of c_2 in (3.49) which are swallowed within $C = -4(c_2)^2$.

At $C = 0$ the RPS is not Hausdorff: the parabolic sector has no disjoint neighborhood with the $(A = 0, C = 0)$ -part of the RPS. Thus there will be no unique way to connect the qualitatively different sectors $C > 0$ and $C < 0$ of the quantum theory.

As indicated previously the RPS above agrees also with the cylindrical factor spaces obtained from the Penrose diagrams Fig. 5: n counts the number of blocks before the identification and the monodromy matrices are phase space analogues of the 'tilt' and the 'circumference' found as the second gauge independent variable beside C in the hyperbolic and elliptic sector, respectively.

The simplest possibility to bring any of the representatives above into a form corresponding to a nondegenerate metric is provided by the gauge transformation $\exp(x^0 T^+) = 1 + x^0 T^+$. (This is true except for $A \equiv 0$ where two transformations are necessary). As a byproduct we find global charts for the Penrose diagrams Fig. 5 and its cylindrical factor spaces in this way.

To get some understanding of the $(e = 0)$ -structure of the orbits, let us parametrize a general $\widetilde{SL}(2, \mathbb{R})$ -element $g(x)$ as follows:

$$g(x) = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}, \quad (3.52)$$

where α , ψ , and χ are arbitrary, in x^1 *periodic* functions of x . That this is a true parametrization can be seen by noting that the group $SL(2, \mathbb{R})$ could be defined as the group of basis transformations in a two-dimensional vector space which leaves the area between two basis vectors invariant; by the first transformation one can change the angle between the two vectors, the second one rotates them, and the third one allows to bring one of the two vectors to any given length. The transition from $\mathcal{G}_{SL(2, \mathbb{R})}$ to $\mathcal{G}_{\widetilde{SL}(2, \mathbb{R})}$ is performed when we excluded quasiperiodic functions ψ .

Now one has to apply the general $\widetilde{SL}(2, \mathbb{R})$ -gauge transformation to any of the representatives of the RPS. Let us do this at the example of the elliptic sector. We can set $\alpha \equiv 0$ for our purposes, since the third transformation corresponds to a Lorentz transformation in the gravity frame bundle and hence it leaves $e^- \wedge e^+ = ed^2x$ unchanged. We then obtain:

$$\begin{aligned} (A^{ell})^g &= (\vartheta dx^1 + d\psi) \begin{pmatrix} \chi & (1 + \chi^2) \\ -1 & -\chi \end{pmatrix} + \begin{pmatrix} 0 & d\chi \\ 0 & 0 \end{pmatrix} \\ (X^{ell})^g &= c_2 \begin{pmatrix} \chi & (1 + \chi^2) \\ -1 & -\chi \end{pmatrix}. \end{aligned} \quad (3.53)$$

As noted above the choice $\psi \equiv 0$, $\chi := x^0$ yields a nondegenerate solution (for $\vartheta \neq 0$): With (3.43) we find $\varepsilon \equiv e^- \wedge e^+ = d\omega = -2\vartheta d^2x$. Let us now analyze the more general transformation provided by

$$\chi = r(x^0) \cos(kx^1), \quad \psi = r(x^0) \sin(kx^1), \quad k \in \mathbf{N}_0, \quad (3.54)$$

where r is some function of x^0 . We then find that $\varepsilon = d\omega = r'[\vartheta \cos(kx^1) + kr]d^2x$. So the resulting metric and zweibein will be nondegenerate on all of the cylinder, iff r is a strictly monotonic function which, for $k \neq 0$, is bounded by $|\vartheta|/k$ from below. A possible choice is, e.g., $r(x^0) := \exp(x^0) + 2|\vartheta|$.

Despite the fact that the gauge transformations (3.52, 3.54) with $\alpha \equiv 0$ and r chosen as above are smoothly connected to the unity for arbitrary value of k , the solutions $(A^{ell})^g$ are gravitationally inequivalent for different values of k . To prove this let us choose a loop \mathcal{C} running around the cylinder once. Under the restriction $\det g = -(\det e)^2 \neq 0$ the components of the zweibein (e_0^+, e_1^+) induce a map $\mathcal{C} \sim S^1 \rightarrow \mathbf{R}^2 \setminus \{0\}$ characterized by a winding number (not depending on the choice of \mathcal{C}). Solutions with different winding numbers cannot be transformed into each other by gravitational symmetries, since they are separated by solutions with $\det e = 0$. For different values of k the solutions (3.53) have different winding numbers, which proves our assertion.

This result generalizes to the other sectors of the theory: Solutions which are gauge equivalent in the $\widetilde{SL}(2, \mathbf{R})$ gauge theory are not equivalent in the gravity theory (as defined in item **2** at the beginning of this section), if they have different winding number. Having found this inequivalence between the factor spaces **1** and **2**, we will not be interested in investigating the latter any further. In particular we will not factor out the large gauge transformations of the gravity theory; since the group of diffeomorphisms and local Lorentz transformations consists only of a finite number of components not smoothly connected to each other,¹⁰ the inequivalence will not be removed by them.

The winding number defined above is related to the kink number as defined in [51] by means of 'turn arounds' of the light cone along noncontractible loops. More precisely, winding number k corresponds to kink number $2k$. (Odd kink numbers [51] characterize solutions which are not time orientable. Such solutions are not considered here).

All the kink solutions found above are geodesically incomplete. The prototype is provided by $k = 1$ since the solutions with $k > 1$ are k -fold coverings of it. For $k = 1$ it is helpful to regard r and x^1 as polar coordinates for hypothetical cartesian coordinates χ and ψ . With the observation that $X^3 = 2c_2\chi$ and an analysis of the null extremals it can be seen that the $(k = 1)$ -solution

¹⁰They differ by x^0 - and x^1 -reflection on the space time manifold and by parity transformation and time reversal in the Lorentz bundle.

(3.53) is the result of cutting out some piece from the Penrose diagram Fig. 5c including some part of one boundary; after the cutting procedure the open ends are identified again. More details shall be provided in [16].

Such a cutting procedure can be performed with any Penrose diagram. Taking the corresponding k -fold covering will yield a $2k$ -kink solution. Thus such solutions exist not only for the Jackiw-Teitelboim, but also for the other models considered in this chapter. By construction their maximal extension is geodesically incomplete,¹¹ providing however gravitationally inequivalent nondegenerate C^∞ solutions with cylindrical topology.

It could be regarded as an advantage of the Hamiltonian formulation with constraints (2.22) that the geodesically incomplete ($k \neq 0$)–solutions are automatically identified with the ($k = 0$)–solution. However, since also the latter are not complete for any choice of C (cf. the discussion in the previous section), this ‘advantage’ seems rather accidental. One of the lessons to be drawn from the analysis of this section is: Equivalence of symmetry transformations only up to $\det g = 0$ in general is of relevance for the factor spaces, and thus for the quantum theory, even if finally all degenerate solutions are excluded.

3.6 Considerations on the Issue of Time

All the models considered within this work can be reduced to quantum systems of finitely many topological degrees of freedom. Thus the question arises: Can such models serve as toy models for a quantum theory of four dimensional gravity? We hope to have convinced the reader within the last sections that this is the case with respect to some technical questions arising in any theory of quantum gravity. It is the purpose of the present section to show that also an illustrative treatment of conceptual questions is possible. We will focus on the so-called ‘problem of time’ [15] of quantum gravity, i.e. the question of how to find any dynamics within such a theory as the standard Hamiltonian vanishes on all (physical) quantum states.

¹¹Except for diagrams such as $G4$ of Fig. 7.

For this purpose we study the example of R^2 -gravity with Minkowski signature coupled to $SU(2)$ Yang Mills. The Lagrangian of this system is

$$L = \int_{S^1 \times \mathbf{R}} \left[\frac{1}{8\beta^2} R_{ab} \wedge *R^{ab} + \frac{1}{4\gamma^2} \text{tr}(F \wedge *F) \right] \quad (3.55)$$

where the Hodge dual operation is performed with the dynamical metric used to define also the torsionless curvature two-form $R_{ab} = \varepsilon_{ab} d\omega(e)$, and the trace is taken in some representation of $su(2)$. We may, e.g., choose $T_i = \sigma_i/2i$ and use the metric $-2\text{tr}T_i T_j = \delta_{ij}$ to lower and raise Lie algebra indices. Rewriting (3.55) by means of Cartan variables in a Hamiltonian first order form, it becomes

$$L_H = \int_{S^1 \times \mathbf{R}} B_a D e^a + B_3 d\omega + \text{tr}(EF) - [\beta^2(B_3)^2 - \gamma^2 \text{tr}(E^2)] \varepsilon \quad (3.56)$$

where we have chosen $E = E^i T_i$ to denote the 'electric fields' conjugate to the $SU(2)$ -connection one-components A_1 , and the B 's are the conjugates to the spin connection ω_1 and the zweibein one-components $e_1^a \equiv (e_1^-, e_1^+)$.

B and E together can be interpreted as coordinates X in a six-dimensional target space N with an appropriate Poisson structure defined on this space. In the present case it is, however, simpler to regard B and E as coordinates for two three-dimensional Poisson structures. $\text{tr}(E^2) = -E_i E_i/2$ may be seen to be a Casimir function of the six- as well as of the two three-dimensional Poisson structures. Thus on-shell it is a constant. S_H is the sum of an $SU(2)$ - EF -theory (2.25) (up to a factor -2) and an action (3.3) with $V = \beta^2(B_3)^2 + \gamma^2 E_i E_i/2$. So one first may solve the *unmodified* $su(2)$ Gauss law (on the classical as well as on the quantum level) and then is left with an ordinary R^2 -gravity theory as studied already before with an effective cosmological constant $\Lambda = -\gamma^2 E_i E_i/2\beta^2$. The coupling between the gravity and the Yang-Mills system is thus seen to be quite 'minimal', but of course not zero.

From the point of view of the field content and the structure of the action, (3.55) is an obvious two-dimensional analogue of the gravity-Yang-Mills system in four dimensions. From the technical point of view it is incomparably simpler. This is precisely what one expects from a model to develop

and/or test conceptual ideas. Although the 'problem of time' arises already in a theory of pure gravity as well, we have chosen to incorporate also the Yang-Mills part in the action. One of the reasons for doing so is that at any point of the considerations we can 'turn off' the gravity curvature by means of the limit $\beta \rightarrow 0$. We are then basically left with a pure Yang-Mills system. In its ordinary formulation the latter, however, has a nonvanishing Hamiltonian and thus a meaningful Schrödinger equation, which should be somehow reproduced in the gravity flat limit. The coefficients in (3.55), where β and γ are understood to be real, have been chosen so as to avoid technical complications as far as possible: In particular there will be no discrete indices within the wave functions arising from the gravity sector, since Λ is effectively negative (cf. Fig. 1 and the discussion in the previous sections).

In explicit terms the constraints following (naturally) from L_H are

$$G_a = \partial B_a + \varepsilon^b_a B_b \omega_1 + \varepsilon_{ab} [-\beta^2 (B_3)^2 + \gamma^2 \text{tr} E^2] e_1^b, \quad (3.57)$$

$$G_3 = \partial B_3 + \varepsilon_b^a B_a e_1^b, \quad (3.58)$$

beside the unmodified $SU(2)$ Gauss law $G \approx 0$. We will not attempt to reformulate these constraints so as to possibly cure the global deficiencies of them with respect to diffeomorphisms noted at the end of the previous section. Instead we proceed with a straightforward quantization.

There are two independent Dirac observables as functions of the momenta:

$$\begin{aligned} q_1 &= \frac{-1}{\pi} \oint \text{tr}(E^2) dx^1 \equiv \frac{1}{2\pi} \oint E_i E_i dx^1 \\ q_2 &= \frac{1}{2\pi} \oint [(B)^2 - \frac{2}{3} \beta^2 (B_3)^3 + 2\gamma^2 \text{tr}(E^2) B_3] dx^1, \end{aligned}$$

where $q_s \equiv \oint C^s dx^1 / 2\pi$ and C^1, C^2 are (the) two Casimir functions of the target space Poisson structure. The corresponding level surfaces have topology $S^2 \times \mathbf{R}^2$ for $q_1 \neq 0$ and \mathbf{R}^2 for $q_1 = 0$.¹² This gives rise to the quantization condition (cf. end of sec. 2.4): $q_1 = n^2/4, n \in \mathbf{N}_0$. Thus the physical wave functions take the form

$$\Psi = \exp \left(\frac{i}{\hbar} \oint (E_3 \partial \varphi \pm \ln B_{\mp} \partial B_3 dx^1) \right) \Psi_0(n, q_2), \quad q_2 \in \mathbf{R}, \quad (3.59)$$

¹²Within the latter level surface the origin is an integral surface by itself. We will in the following disregard this small complication.

having written the phase factor in some local target space coordinates with $\tan \varphi \equiv (E_2/E_1)$.

Expanding the physical wave functionals in terms of eigenfunctions $|n\rangle$ of q_1 , we may write alternatively

$$\Psi = \sum_{n=0}^{\infty} \exp\left(\frac{i}{\hbar} \pm \oint \ln B_{\mp} \partial B_3 dx^1\right) \tilde{\Psi}_n(q_2) |n\rangle. \quad (3.60)$$

This makes contact with our previous observation: The coefficients in the above expansion are R^2 -gravity wave functions for the respective cosmological constant $-\gamma^2 n^2/8\beta^2$.

The inner product with respect to q_2 is determined by the hermiticity requirement on (cf. (3.9))

$$p_2 = -\frac{1}{2} \oint \frac{e_1^{\pm}}{B_{\mp}} dx^1, \quad (3.61)$$

the Dirac observable conjugate to q_2 : as noted already previously, p_2 acts as the usual derivative operator on $\tilde{\Psi}_n$, thus leading to the ordinary Lebesgue measure dq_2 . Of course the hermiticity of q_1 leads to the orthogonality of $|n\rangle$ for different n .

We end up with the Hilbert space \mathcal{H} of an effective two-point particle system with nontrivial phase space topology (giving rise to the discrete spectrum of q_1). As a basic set of operators acting in \mathcal{H} we could use q_2, p_2, q_1 , and $tr[\mathcal{P} \exp(\oint A_1 dx^1)]$. From the latter one may construct a ladder operator: $n \rightarrow n + 1$.

All operators acting in \mathcal{H} are thus found to be expressible in terms of q_2, p_2 , and the number and ladder operators. However, we do not have an operator such as $g_{\mu\nu}(x^\mu)$. Following, furthermore, any textbook on elementary quantum mechanics, the next step in the quantization procedure would be to introduce an evolution parameter 'time', which we will call τ , and to require the wave functions to evolve in this parameter according to the Schrödinger equation. In the present case, however, the Hamiltonian following from (3.56) is a combination of the constraints,

$$H = - \oint [e_0^a G_a + \omega_0 G_3 + tr(A_0 G)], \quad (3.62)$$

so that the naive Schrödinger equation becomes meaningless.

Both of these items, the nonexistence of space-time dependent quantum operators as well as the apparent lack of dynamics, are correlated and they are not just a feature of the topological theory (3.55). Also in four dimensional gravity the quantum observables are some (not explicitly space-time dependent) holonomy equivalence classes and the Hamiltonian vanishes when acting on physical wave functions [1]. Diffeomorphisms are part of the symmetries of any gravity theory; as a consequence the Lie derivative into any 'spatial' direction can be found to equal the Hamiltonian vector field of some linear combination of the constraints (in our case $\mathcal{L}_1 = e_1^a G_a + \omega_1 G_3 + tr A_1 G$, cf. Eq. (2.19)), whereas, on shell, x^0 -diffeomorphisms will be generated by the Hamiltonian H . Thus, although 4D gravity has local degrees of freedom, any of its (uncountably many) Dirac observables will be also space-time independent.

To orientate ourselves as of how to introduce quantum dynamics within such a system, let us have recourse to the simple case of a nonrelativistic particle (NRP). As is well known, any Hamiltonian system can be reformulated in time reparametrization invariant terms. In the case of the NRP,

$$\int (p \frac{dq}{dt} - \frac{p^2}{2}) dt = \int (p \dot{q} - \frac{p^2}{2} \dot{t}) d\tau, \quad (3.63)$$

the equivalent system has canonical coordinates $(q, t; p, p_t)$ and the 'extended' Hamiltonian is proportional (via a Lagrange multiplier) to the constraint $K = p^2/2 + p_t \approx 0$. Quantizing this system, e.g., in the coordinate representation, we observe that the implementation of the constraint $K\psi(q, t) = 0$ is equivalent to the Schrödinger equation of the original formulation, if one reinterpretes the canonical variable t as evolution parameter τ . Therefore, given this formulation of the NRP or similarly of any other system, the postulate of a Schrödinger equation within the transition from the classical to the quantum system becomes superfluous; rather it is already included within the Dirac quantization procedure in terms of a constraint equation.

The identification $t = \tau$ above can be looked upon also as a gauge condition with gauge parameter τ . This interpretation is helpful for the quantization of the parametrization invariant NRP in the momentum representation

$\psi(p, p_t)$, in which case the space of physical wave functions is isomorphic to the space of functions of the Dirac observable p . The gauge condition $\bar{K} \equiv t - \tau = 0$ provides a perfect cross section for the flow of K . Thus it is possible to determine any phase space variable in terms of the Dirac observables p , $Q = q - pt$, as well as the gauge fixing parameter τ . Interpreting τ as a dynamical flow parameter 'time', the obtained evolution equations for p and q , transferred to the quantum level as $q(\tau) = i\hbar d/dp + \tau p$, $p(\tau) = p$, become equivalent to the Heisenberg evolution equations of the parametrized NRP.

The operator $q(\tau)$ above corresponds to a measuring device that determines the place of the particle at time τ . A measuring device that determines the time t at which the particle is at a given point $q = q_0$, on the other hand, corresponds to the alternative gauge condition $\tilde{K} \equiv q - q_0 = 0$. \tilde{K} provides a good cross section only for $p \neq 0$. Ignoring this subtlety, e.g. by regarding only wave functions with support at $p \neq 0$, the (hermitian) quantum operator for such an experiment is $t(q_0) = -i\hbar [(1/p)d/dp - (1/2p^2)] + q_0/p$. In this second experimental setting Heisenberg's 'fourth uncertainty relation' between time t and energy $p^2/2 \sim -p_t$, usually motivated only heuristically, becomes a strict mathematical equation. We learn that different experimental settings are realized by means of different gauge conditions, and, at least in principle, vice versa.

The wave functions of (3.56) are basically functions of the Dirac observables, although part of the latter became discretized in the quantum theory. Transferring the ideas above to the gravity system, we should find gauge conditions to the constraints (3.57, 3.58). (It will not be necessary to gauge fix also the $su(2)$ Gauss law G). As such we will choose

$$\partial B_+ = 0, \quad B_3 + \tau B_+ = 0, \quad e_1^- = 1. \quad (3.64)$$

It is somewhat cumbersome to convince oneself that this is indeed a good gauge condition. However, for $q_1 \neq 0$ it provides even a globally well-defined cross section.

[One possibility to check the obtainability of (3.64) is to carefully analyze the corresponding Faddeev matrix, taking into account that the constraints

are not completely independent due to (2.34). This (infinite dimensional) matrix turns out to be nondegenerate, iff $B_+ \oint e_1^- dx^1 \neq 0$. For $q_1 \neq 0$ any gauge orbit in the loop space contains a representative fulfilling this condition, which suffices to prove the assertion since the space of gauge orbits is connected in the case under study (no quantum number n).]

The gauge conditions (3.64) together with the constraints allow to express all gravity phase space variables in terms of Dirac observables. In this way one obtains evolution equations such as

$$B_-(\tau) = -\frac{1}{2\pi}p_2q_2 - \frac{\gamma^2}{2}q_1\tau - \frac{\beta^2\pi^2}{3(p_2)^2}\tau^3, \quad B_+(\tau) = \frac{-\pi}{p_2}. \quad (3.65)$$

Antisymmetrizing this with respect to q_2 and p_2 , (3.65) can be taken as an operator in the Hilbert space \mathcal{H} defined above.¹³ Similarly one finds $g_{11}(x^0) = 2e_1^+(x^0) = -p_2B_-(x^0)/\pi$, ($x^0 \equiv \tau$), which now, up to operator ambiguities, becomes a well defined operator in our small quantum gravity theory, too.

Requiring that the τ -dependence of (3.64) is generated by the Hamiltonian H , the gauge conditions determine also the zero components of the zweibein and the spin connection. Actually, one zero mode of these Lagrange multiplier fields e_0^a, ω_0 remains arbitrary as a result of the linear dependence (2.34) of the constraints G_i (cf. also [20]). Requiring this zero mode to vanish as a further gauge condition, one finds $e_0^+ = 1$ and $e_0^- = \omega_0 = 0$. In other gauges the Lagrange multipliers can become also nontrivial quantum operators. Furthermore, it is a special feature of the chosen gauge (adapted to the Killing direction $B_3 = \text{const}$) that the obtained operators are x^1 -independent. Again different choices of gauge conditions are interpreted as corresponding to different types of questions or measuring devices.

The alternative procedure to reintroduce time within the quantum theory of the parametrization invariant NRP was the direct implementation of the gauge within the wave functions. For this it was decisive that the initially

¹³The elementary procedure above coincides with the use of Dirac brackets for τ -dependent systems (in which case one extends the symplectic form by $d\tau \wedge dp_\tau$); this explains also B_- and B_+ do not commute anymore.

chosen polarization of the wave functions, $\psi(q, t)$, contained the phase space variable subject to the gauge. To implement (3.64) analogously within the gravity theory under consideration, we Fourier transform (3.60), multiplied by $\delta[\partial C^2]$, with respect to $B_-(x^1)$. The result is

$$\exp \left(\frac{i}{\hbar} \oint [E_3 \partial \varphi + \frac{\partial B_+ B_3 + [\frac{\beta^2}{3}(B_3)^3 - \gamma^2 \text{tr}(E^2) B_3] e_1^-}{B_+}] dx^1 \right) \Pi_{x^1} \left(\frac{\text{const}}{B_+} \right) \hat{\Psi}_n(p_2), \quad (3.66)$$

in which $\hat{\Psi}_n$ is the Fourier transform of the ordinary function $\tilde{\Psi}_n$.

Eq. (3.66) is in agreement with the general solution of the quantum constraints in a (B_+, B_3, e_1^-, E) representation, if we stick to the operator ordering resulting from the Fourier transformation of (2.33). Putting, on the other hand, all derivative operators in the quantum constraints to the right to start with, we again find no quantum anomalies in the constraint algebra. However, the latter operator ordering violates the conditions (2.34) and thus leads inevitably to an empty kernel of the constraints.¹⁴

In the gauge (3.64) the quantum wave functions (3.60) take the form

$$\Psi = \sum_n \exp \left[\frac{-i}{\hbar} \left(\frac{\gamma^2 n^2}{8} \tau + \frac{\beta^2 \pi^2}{3 p_2^2} \tau^3 \right) \right] c_n(p_2) |n\rangle, \quad (3.67)$$

where we have absorbed the divergent factor of (3.66), being a function of p_2 , into $c_n(p_2)$.

At this point it is worthwhile to perform the limit $\beta \rightarrow 0$. In some sense (3.55) with $\beta = 0$ is the parametrization (i.e. diffeomorphism) invariant formulation of the usual Yang Mills theory on the cylinder (with rigid Minkowski background metric). If we ignore the p_2 dependence of c_n for a moment, (3.67) with $\beta = 0$ indeed coincides with the time evolution generated by the (nonvanishing) Yang Mills Hamiltonian $-\gamma^2 \oint \text{tr} E^2 dx^1 \equiv \gamma^2 \pi q_1$. This agreement gives support to the method used to derive (3.67).

The reason for the p_2 -dependence of c_n is due to the fact that in the formulation (3.56) with $\beta = 0$ the metric induced circumference of the cylinder

¹⁴It would be interesting to see, if a similar mechanism is responsible for the apparent lack of physical states in four-dimensional $N = 1$ supergravity [52].

der became a dynamical variable (on shell one has $p_2 \propto \oint_{B_3=const} \sqrt{g_{11}} dx^1$). Within (3.64) one finds $-\oint G_+ \sim H$ to effectively implement the Schrödinger equation corresponding to (3.67). The effective Hamiltonian acting on $c_n|n\rangle$ is $-(\gamma^2/2) \oint tr E^2 dx^1 - \beta^2 \pi^2 \tau^2 / p_2^2$. Thus generically the above procedure yields time dependent Hamiltonians.

In the case of the unparametrized NRP the 'Heisenberg picture' and the 'Schrödinger picture' approach to introduce dynamics are obviously equivalent. Straightforward equivalence of these two approaches was established also for the gauge $X_+ = 1$, $X_3 = \tau$, $\partial e_1^- = 0$ in [20].¹⁵ It is, however, not quite clear if or in how far the same is true also for the present incorporation of the gauge conditions (3.64). Further investigations into this direction, analyzing the subject also from a more abstract point of view, are desirable.

The strategies developed at the example of a NRP to resolve the 'issue of time' within a quantum theory of gravity produced, however, quite sensible results for the toy model (3.55). But they relied heavily on either the knowledge of all Dirac observables or on some specifically chosen polarization. To cope with the considerable technical difficulties of a quantum theory of four-dimensional gravity, it might be worthwhile to extend the applicability of the method.

One way to do so within our model is to allow for equivalence classes of wave functions coinciding at $\partial Q_2 = 0$, the latter condition being enforced within the inner product [20]. In this way one can, e.g., implement the gauge condition $\partial e_1^- = 0$ as an operator condition in the B -polarization of the wave functions as well. Still, however, a straightforward implementation of $\oint e_1^- = const$ seems inadmissible also then in this polarization.

Given the open ends which may be found in this section, we still hope to have convinced the reader that nontrivial (quantum) dynamics in a theory of gravity corresponds, in one way or the other, to the choice of gauge conditions which break the diffeomorphism invariance.

¹⁵There the analysis was performed for the KV-model (1.4), but is valid in an obvious way also here.

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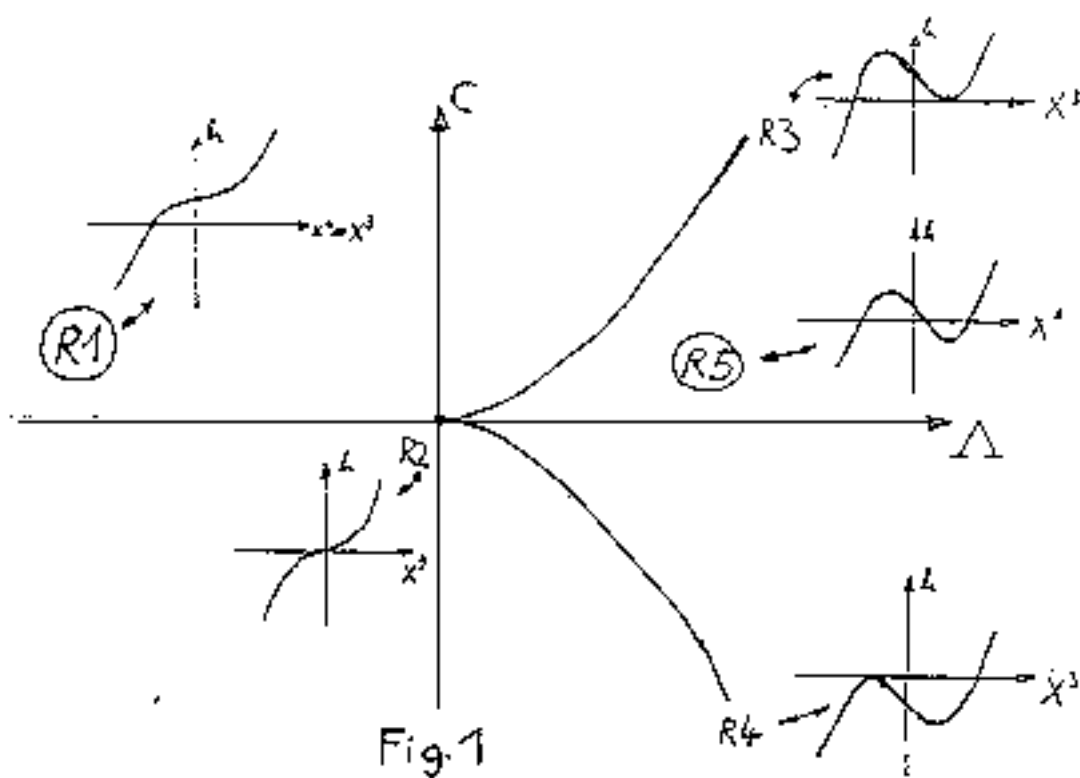
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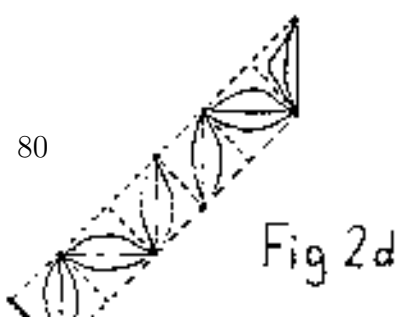
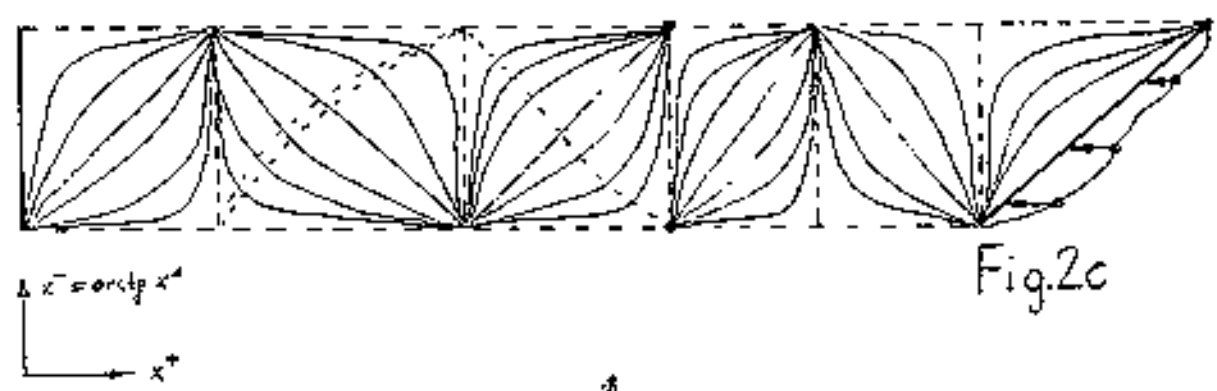
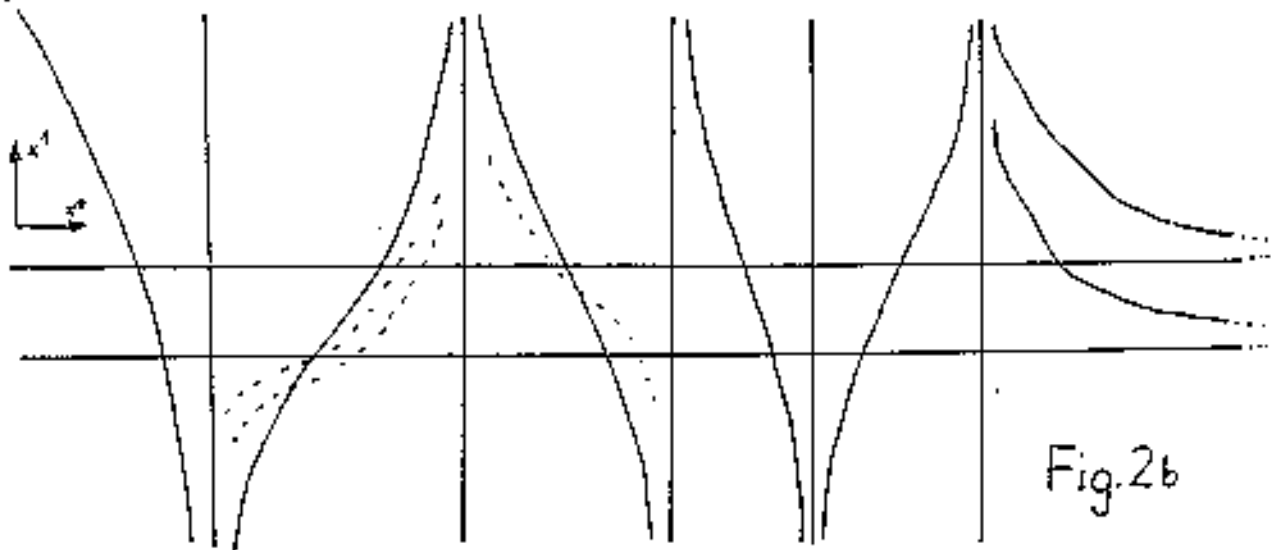
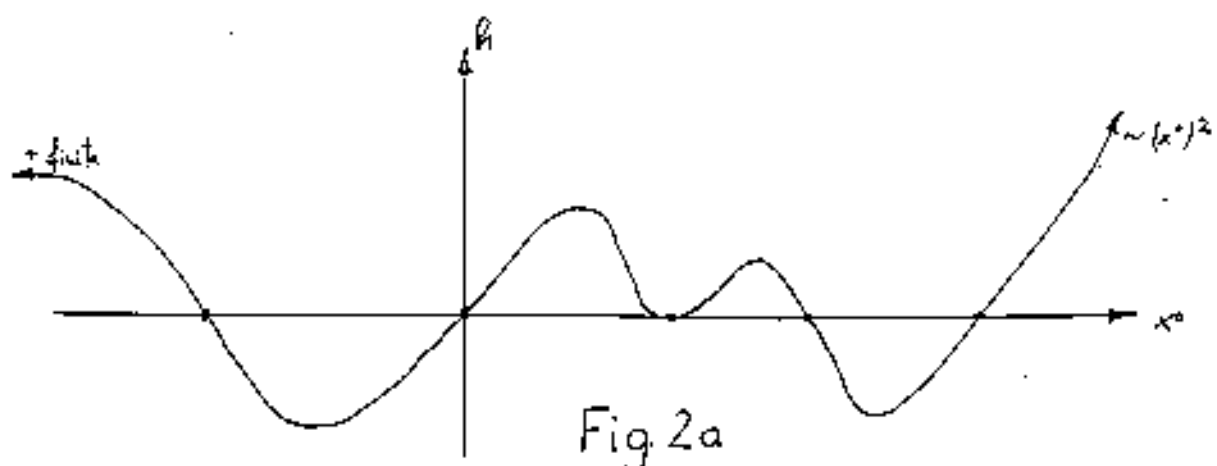
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Figures





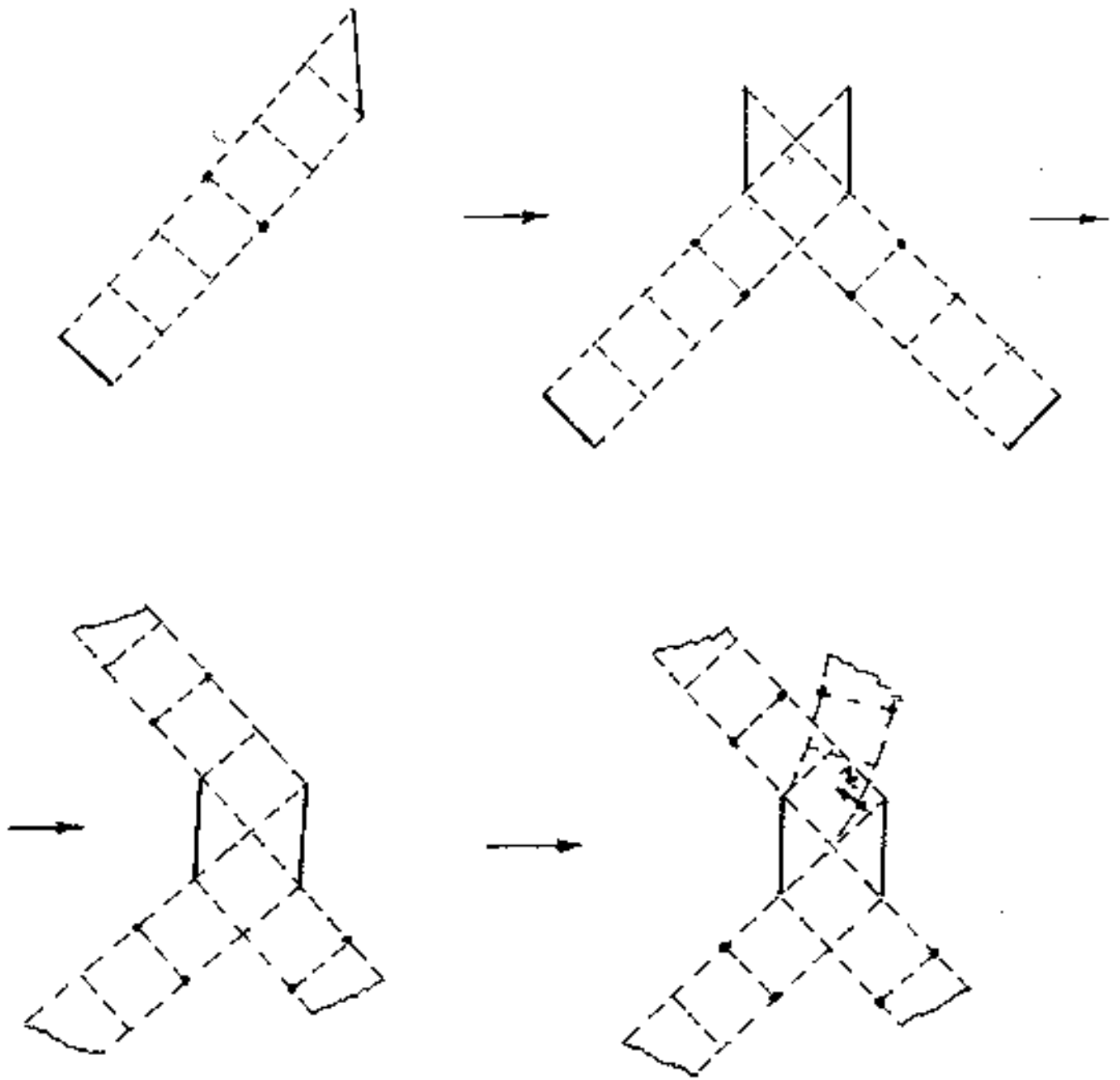


Fig. 3

Fig. 4

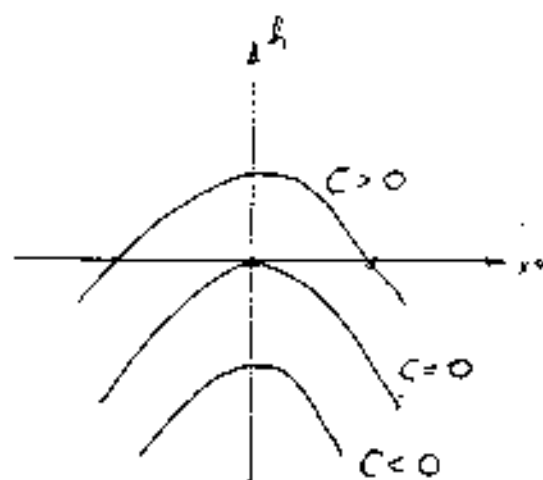


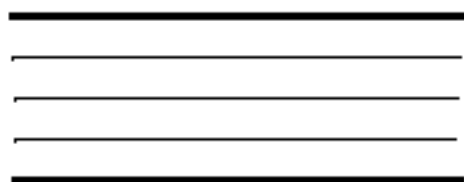
Fig. 5a



Fig. 5b



Fig 5c



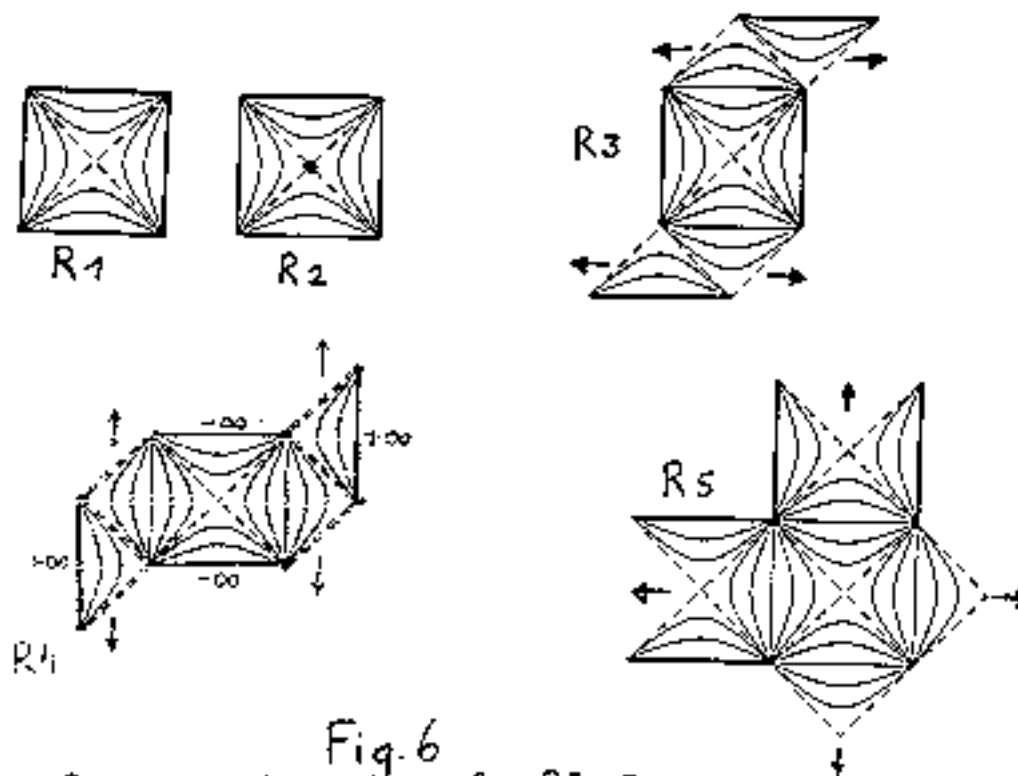


Fig. 6
Penrose diagrams for R^2 -Gravity

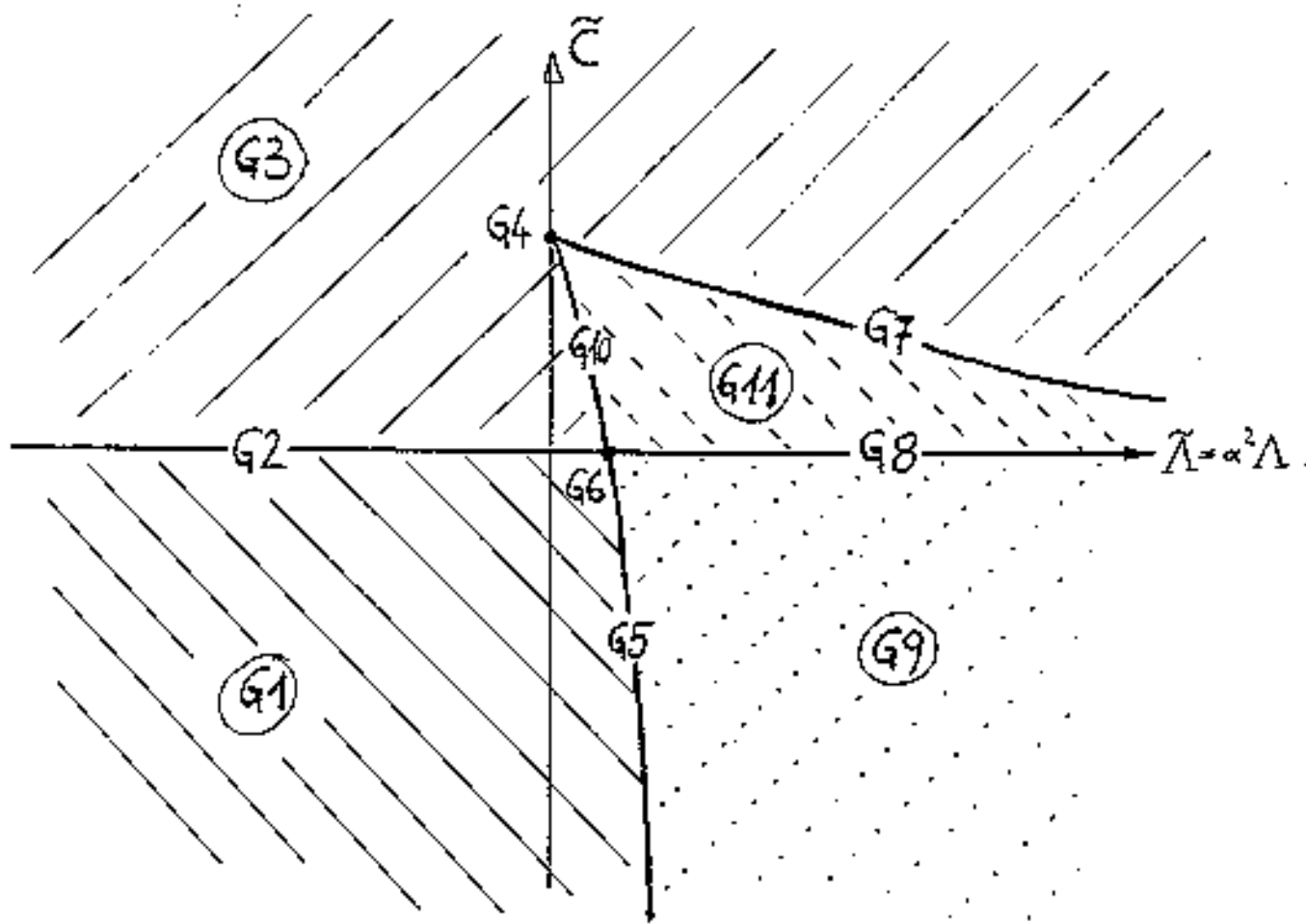


Fig. 7

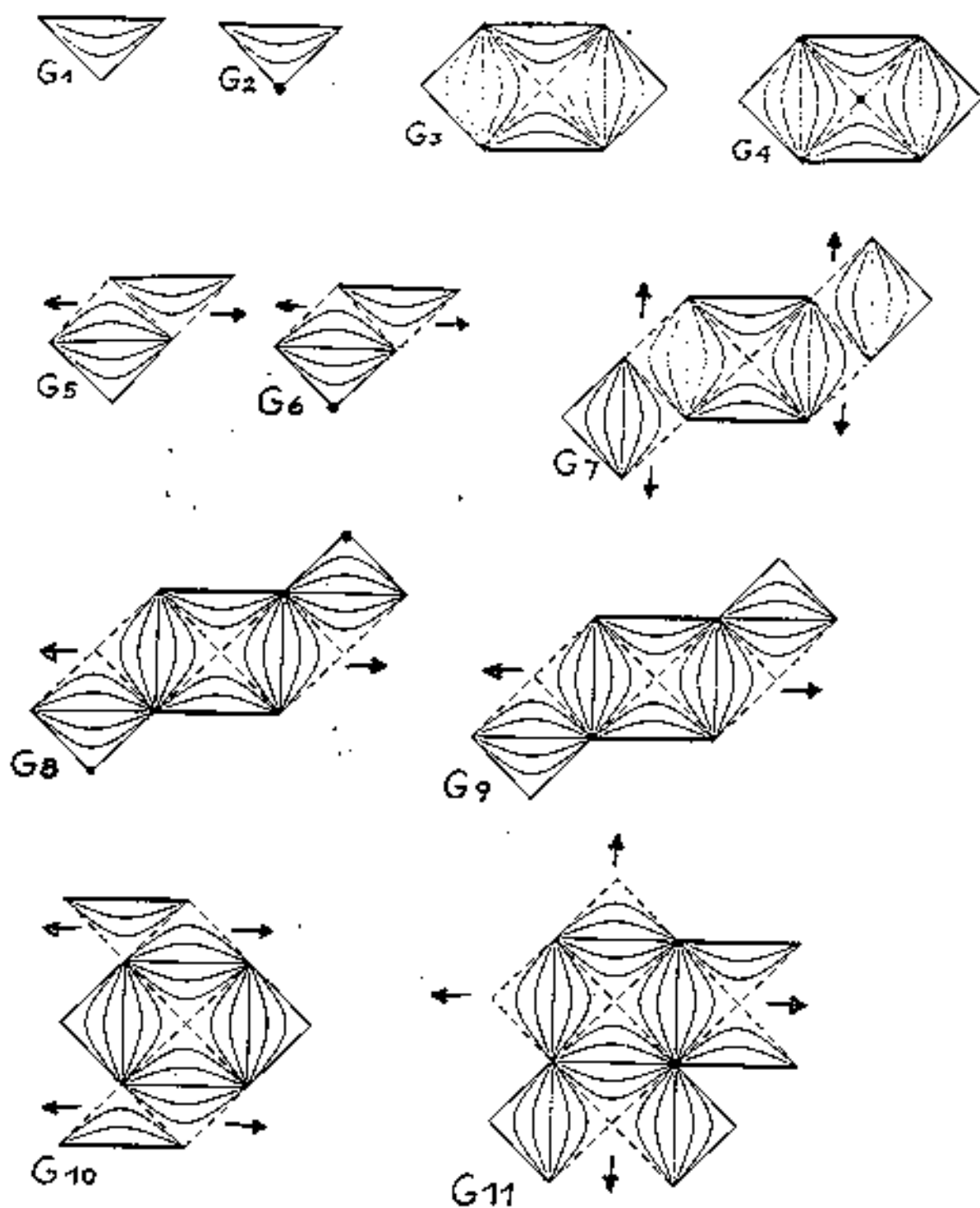


Fig. 8
 Penrose diagrams₈₅ for the Katanaev-Volovich-Model